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P. ABBOTT, B.A., C. S. JACKSON, M.A.
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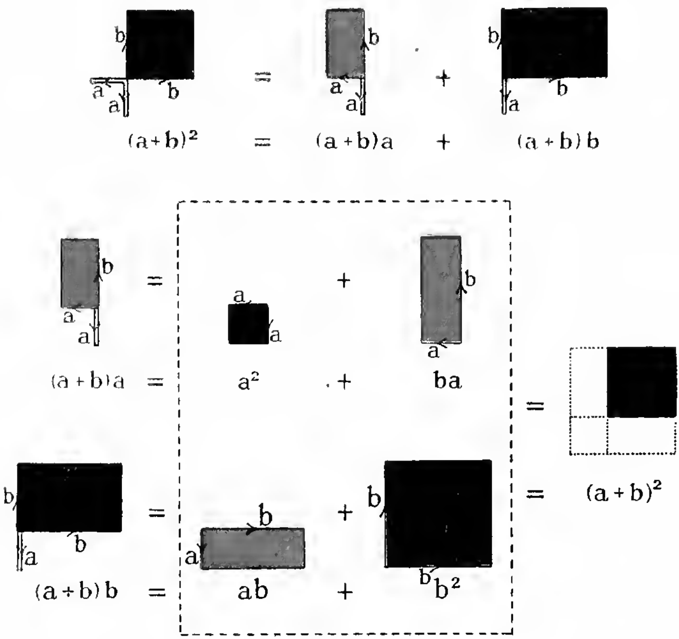
$(a + b)^2 = a^2 + 2ab + b^2.$
a negative, b positive ; a numerically less than b.

$$
\begin{aligned}
(a + b)(a + b) &= a^2 + ba \\
&\quad\; + ab + b^2 \\
&= a^2 + 2ab + b^2.
\end{aligned}
$$



[See Ex. XXXI, p. 183.]

Longmans' Modern Mathematical Series

EXERCISES IN ALGEBRA

(INCLUDING TRIGONOMETRY)

BY

T. PERCY NUNN, M.A., D.Sc.

VICE-PRINCIPAL OF THE L.C.C. LONDON DAY TRAINING COLLEGE (UNIVERSITY OF LONDON); FORMERLY SENIOR MATHEMATICAL AND SCIENCE MASTER WILLIAM ELLIS SCHOOL

PART I.

WITH DIAGRAMS

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1913

NOTE.

THE exercises in this volume are intended to supply the materials for a course in Algebra to be completed at about the age of sixteen. In addition to the subjects usually required in the Entrance Examinations of the Universities it contains exercises upon logarithms and the elementary parts of trigonometry, and a simple introduction to the methods of the differential and integral calculus.

A discussion of the principles upon which the author has selected and presented the subject-matter of the exercises will be found in the companion volume on "The Teaching of Algebra". That volume contains also suggestions as to the order in which the exercises should be taken, and a full statement of the preparatory teaching pre-supposed in each. It includes a similar treatment of the ground covered in "Exercises in Algebra," Part II, in which the course commenced in the present book is completed.

The author will be greatly obliged if readers who detect errors or obscurities in the examples or the answers will kindly direct his attention to them.

LONDON DAY TRAINING COLLEGE
(UNIVERSITY OF LONDON),
July, 1913.

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SECTION I.

NON-DIRECTED NUMBERS.

EXERCISE I.

THE "SHORTHAND" OF ALGEBRA.

Express in the "shorthand" of Algebra the rules mentioned in Nos. 1-12.

1. The rule for calculating the area of an oblong floor when you know its length and its breadth.

2. The rule for calculating the breadth of an oblong room when you know its length and the area of the floor.

3. The rule for calculating the cost of a number of things when you know the price of each. Also the rule for calculating the price of a single article when you know that a certain number of them cost so much. (Let $n \equiv$ "the number bought," $C \equiv$ "the total cost," $p \equiv$ "the price of each thing".)

4. The rule for calculating the cost (C) of a certain number of things (N) when you know how much (c) another number of things (n) costs.

5. The rule for finding the number of pence in a given number of shillings. (Let $p \equiv$ "the number of pence," $s \equiv$ "the number of shillings".)

6. The rule for reducing pence to shillings.

7. The rules for reducing pounds to shillings and pounds to pence. (Let $L \equiv$ "the number of pounds".)

8. The rules for reducing shillings to pounds and pence to pounds.

9. The rule for calculating the total area of three oblong rooms each of the same length and the same breadth. The same rule written to suit any number (n) of rooms.

10. The rule for calculating the area of a square room. (Let $s \equiv$ "the length of the side".) The rule for the area of a number of such rooms all of the same size.

11. The rule for calculating the volume of an oblong room (i) given the area of the floor (A) and the height (h); (ii) given the length, breadth, and height.

12. The rule for calculating the depth of water in an oblong

cistern (i) when you know the volume of water (V) and the area of the bottom of the cistern; (ii) when you know the volume of water, the length and the breadth of the cistern.

13. A number of persons subscribed equal amounts to send the children in a certain school away for a holiday. Write a formula for the amount subscribed by each person, given the number of children and the expense of sending each child away. Use the symbols of No. 4.

14. How must the last formula be changed if the expense per head is reckoned in shillings and the subscription in pounds?

15. The rule for making tea is: "One spoonful of tea for each person and one for the pot". Reduce this rule to a formula letting $t \equiv$ "the number of spoonfuls of tea," $p \equiv$ "the number of persons".

16. Write formulæ for the rules used (i) in reducing shillings and pence to pence; (ii) in reducing pounds and shillings to shillings; (iii) in reducing pounds and pence to pence.

17. Give in a formula the Post Office rule for the cost of an inland telegram. (Let $C \equiv$ "the cost of the telegram in pence," $n \equiv$ "the number of words over twelve".)

18. A library charges an entrance fee of half a crown and a subscription of twopence for every book borrowed. Write a formula for the total amount paid (in shillings) for borrowing a given number of books.

19. The following rule for cooking a joint of beef is often given: "Allow a quarter of an hour for every pound and twenty minutes over". Express this rule in a formula for the time in hours needed to cook a joint of a given weight.

20. I am twenty-seven years younger than my father. Write in a formula the rule for finding my age when his age is known. (Let $A \equiv$ "my father's age," $a \equiv$ "my age".)

21. My age is three years less than half my father's age. Write the rule for finding my age, given my father's.

22. A second-hand bookseller bought a number of books for £1 10s. and sold them at 9d. each. Write a formula for his profit (in shillings) after selling a certain number of them (n).

23. A greengrocer buys a number of oranges at a certain price per dozen and sells them at so much each. Write a formula for his profit after selling a certain number. (Let $P \equiv$

" the price of a dozen," $p \equiv$ " the price of a single orange," $N \equiv$ "the number of dozens bought," $n \equiv$ " the number of single oranges sold".) How would you write the formula if there was a loss instead of a gain?

24. Write down as a formula the rule for finding the simple interest for a given number of years on a given sum of money at a given rate of interest per cent per annum.

25. Change the foregoing formula into one for finding the amount of the sum of money instead of merely the interest on it.

EXERCISE II.

GRAPHIC REPRESENTATION.

A.

1. Fig. 1 is a copy ($\frac{1}{3}$ of the actual size) of the traces left by two snails after wandering on a sheet of paper. The numbers indicate their positions after 1, 2, 3, . . . 9 minutes. Make a table of the distances travelled by each snail during each minute. Draw a diagram, in accordance with instructions,

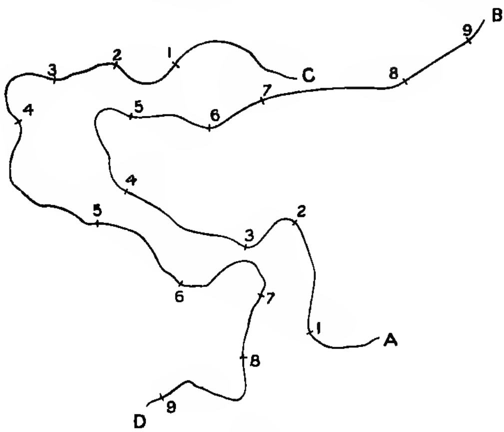


FIG. 1.

showing by vertical lines the distances covered in each minute by the snail which left the trace marked AB. Draw another diagram showing the movements of the other snail.

2. Draw, in the way explained to you, a diagram or graph by which the speeds of the two snails during any given minute can be compared. Which had the greatest average speed during (i) the first, (ii) the third, (iii) the fourth minute? Why is it necessary to say "average speed"?

3. A cyclist travels along the same road in the same direction on two occasions. On each occasion he keeps account of the distances he rides in the first, second, third, . . . hours of his journey. They are given in the following table. Draw, as in No. 2, a graph by which his performances in corresponding hours on the two occasions may be compared:—

Hour . .	1	2	3	4	5	6	7
Miles ridden:—							
1st journey	11	8	$9\frac{1}{4}$	13	$6\frac{1}{2}$	$10\frac{1}{2}$	12
2nd journey	$10\frac{1}{4}$	9	$9\frac{1}{4}$	8	$10\frac{1}{2}$	$11\frac{1}{4}$	$11\frac{3}{4}$

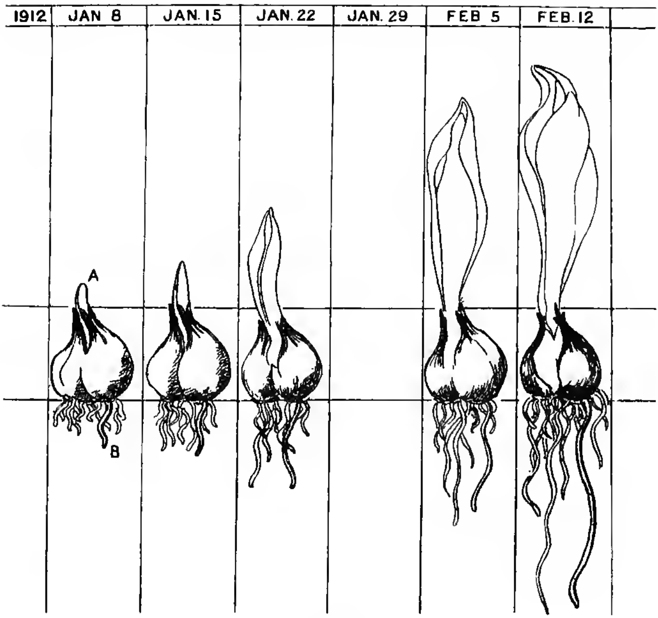


FIG. 2.

4. Fig. 2 is a series of pictures ($\frac{1}{3}$ of the natural size) of a growing tulip. The bulb began to germinate on a Monday morning and the pictures were taken on successive Monday mornings afterwards with the exception of the fourth which was drawn a fortnight after the third. Draw, as instructed,

a diagram composed of vertical lines representing the total length of the plant (from A to B) at each measurement. Draw in its proper place a dotted line representing the length which the plant probably had on the Monday when the measurement was omitted. (*Note.*—The process of supplying in this way a missing observation is called **interpolation.**)

5. Draw a graph (like that of No. 1) showing the *increase* in length of the plant in each week. Is there in this graph anything which serves as a test of the correctness of the interpolation in No. 4?

6. If during one of the hours in No. 3 the cyclist had neglected to note the distance he rode, would it be safe to supply the omission by interpolation?

7. On September 27, 1911, a rod 6 inches high was erected vertically upon a drawing-board and placed in full sunshine. The following table gives the lengths of the shadow of the rod at different times during the day. Draw a horizontal "time-line" (as in No. 5) graduated in hours. At the proper points draw perpendicular lines to represent the various lengths of the shadow. Draw carefully the curve which you would use for finding by interpolation the length of the shadow at times not mentioned in the table.

Time	10.0	10.40	11.15	11.30	11.45
Length of shadow in ins.	8·60	7·79	7·35	7·25	7·22
Time	12.0	12.20	1.0	2.15	
Length of shadow in ins.	7·22	7·30	7·75	9·81	

8. At what time on September 27, 1911, was the shadow of an upright rod shortest? (Note that it is not exactly at 12 o'clock. "Noon" coincides with 12 o'clock only very rarely.) What would the length of the shadow have been at (i) 11 o'clock, (ii) 2.30 p.m.?

9. Divide the base line in No. 7 into half-hour intervals, measuring right and left from the time of noon. Starting with noon find how much the shadow lengthens during each half-hour towards the evening. Make a graph of the half-hourly increase as in Nos. 1 and 5, drawing in the smooth curve which the measurements suggest.

10. Do the same with the half-hourly decrease in the shadows from the morning on till noon. What facts do the two graphs bring out?

B.

Note.—In the graphs of Nos. 4, 5, 7, and 9, the most important thing is the smooth curve. This curve shows how the quantity rises or falls in size, and enables us by "interpolation" to find its probable magnitude at a time when we did not actually measure it. The position of the curve could be fixed without actually drawing the vertical lines if the points where the ends of those lines would come were marked. Moreover, the absence of the vertical lines would make the graph clearer. In the following graphs the vertical lines are not to be drawn. Mark the points where their ends would come if they were drawn, and run the graph smoothly through or among those points. The best way to tell whether you are drawing the graph properly is to hold the paper horizontally close to the eye so that you can look along the curve.

11. A wooden beam is supported horizontally on the edges of two files and is loaded with weights at the point midway between the supports. The following table shows how much a given weight presses the middle of the beam down. Exhibit the results of the observations in a graph. (How much would the beam sink if no weight at all was placed on it? What observation, therefore, will you enter in the graph in addition to those given below?)

Weight in lb.	2	4	6	8	10	12	14	16
Sag in inches	0·3	0·6	0·9	1·2	1·5	1·8	2·1	2·4

12. In No. 11 find the sagging produced in the beam by a load of (i) 5 lb., (ii) 11 lb. Also find what load would make the beam sag (iii) ½ inch, (iv) 2 inches.

13. A marble was allowed to roll down a smooth groove cut in a sloping plank. The following table shows the distances it rolled in 1, 2, 3, . . . seconds. Exhibit these distances in a graph. Use the graph to find how far the marble would roll in (i) 1·5 seconds, (ii) 4·3 seconds. Also use it to find how long the marble would take to roll (iii) 50 cms., (iv) 100 cms.[1] (Read the questions at the end of No. 11.)

Time in seconds	1	2	3	4	5	6
Distance in cms.	8	32	72	128	200	288

[1] This experiment was first performed by the great Galileo about 1638.

14. Draw a graph showing how far the marble rolled in the first, second, third . . . seconds. What information does the graph give with regard to the increase of the average speed of the marble during each second? Find by interpolation how far the marble rolled (i) between 1·7 seconds and 2·7 seconds after starting; (ii) between 3·6 seconds and 4·6 seconds after starting. Answer the same questions by means of the graph of No. 13. Do the answers agree?

Note.—In the foregoing graphs the vertical lines have always represented actual lengths. There is no reason why they should not represent other quantities, for example, money. (If you please you may think of the length of a vertical line as representing the height of a pile of coins.)

15. A furnishing firm advertises the following rates at which furniture may be bought by monthly payments. Draw a graph representing the given rates. What monthly payments should secure furniture to the value of (i) £30, (ii) £40, (iii) £85?

Value of furniture	£10	£20	£50	£75	£100
Monthly payment	6s.	10s.	26s.	35s.	42s.

16. The table gives in degrees the height of the sun above the horizon ("altitude") as measured in London at different times on August 23, 1912. Exhibit the observations in a graph. Use the graph to determine (i) the greatest height reached by the sun during the day; (ii) the time of noon; (iii) the time of sunrise and sunset.

Time	*Alt.*	*Time*	*Alt.*
a.m.	deg.	p.m.	deg.
6.40	15	12.40	49½
7.0	18	1.23	47
7.55	26½	2.15	42
8.43	33½	2.43	38½
9.5	36½	3.15	34½
9.45	41½	4.2	27½
10.27	45½	4.30	23½
11.15	49	4.43	21½
11.26	49½	5.15	16½

Note.—It is often unnecessary to represent the whole of each vertical line. When all these lines are greater than a certain length the excess above this length is all that need be represented. For example, in No. 17 the vertical scale may

begin with the number 20. The lowest 20 inches of each vertical may be supposed to be below the base.

17. A rubber cord, 23·7 cms. long, is hung up vertically and a number of weights are attached to the lower end in succession. The table gives the length of the cord for certain weights. Exhibit the observations in a graph. What weight would make the cord 25 cms. long? How long would it be if 18 gms. were attached?

Weight in gms.	5	10	15	20	25	30
Length in cms.	24·3	24·9	25·5	26·1	26·7	27·3

18. A heavy button was hung at the end of a piece of silk thread and allowed to swing like a pendulum. The table gives the time taken for 100 swings with different lengths of silk. Draw a graph of the results, and use it to find what length of silk would give a pendulum beating seconds. What would be the time of *one* swing when the silk is 2 feet 6 inches long?

Length in feet .	1	2	3
Time of 100 swings	55 sec.	1 min. 18 sec.	1 min. 35 sec.
Length in feet .	4	5	6
Time of 100 swings	1 min. 50 sec.	2 min. 4 sec.	2 min. 15 sec.

19. The "lighting up time" for cyclists and motorists is given in the following table for certain dates in 1912. Draw a graph by which the lighting up time on intermediate days could be determined. When must a cyclist light up on (i) February 12, (ii) May 3? When can the cyclist first ride without a light until 8 o'clock?

Jan. 1	Jan. 31	Mar. 1	Mar. 31	Ap. 30	May 30	June 29
4.58	5.43	6.38	7.29	8.18	9.3	9.19

20. If you look at the back of a Post Office Savings Bank Book you will find the following table. It states the single premium or sum which you must pay in order to receive £100 when you are 60 years of age or to secure £100 for your relatives if you die before that age. The premium depends upon the age at which you insure. Draw a graph and use it to find the premium you would have to pay (i) at 18, (ii) at 32.

Age next birthday.	*Premium.*		
	£	s.	d.
15	41	4	6
20	45	5	0
25	49	5	6
30	53	16	6
35	59	1	6
40	65	2	0
45	72	1	0
50	80	3	6

C.

21. During a very wet season certain fields in the Thames valley were flooded. During the first day of the flood the water covered 2 acres, on the second day it spread over 3 acres more. During the following five days 4·3 acres, 4·8 acres, 3·6 acres, 2·3 acres, 0·7 acres were added in succession to the area covered. Draw a column-graph showing the way in which the area covered by the flood grew from day to day.

22. Convert the column-graph into a graph showing the way in which the area of the flood probably grew from hour to hour. Why must one say "probably"? What area did the water probably cover (i) $1\frac{1}{2}$ days, (ii) $4\frac{1}{2}$ days after the beginning of the flood?

23. In a terminal examination in algebra out of a class of 28 boys the marks given ran as follows. Between 10 and 20 per cent., 2 boys; 20-30 per cent., 2 boys; 30-40 per cent., 3 boys; 40-50 per cent., 4 boys; 50-60 per cent., 5 boys; 60-70 per cent., 5 boys; 70-80 per cent., 4 boys; 80-90 per cent., 2 boys; 90-100 per cent., 1 boy. Draw a column-graph exhibiting these results. What is the use of drawing such a graph? What is the total area of the rectangles?

24. Obtain (i) the ages and (ii) the heights of all the boys or girls in your form. Divide the greatest differences of age and height each into the same number of steps, and find (as in the preceding example) how many boys or girls are included in each step. Draw two column-graphs, one to show the distribution of ages and the other the distribution of heights in your form. Compare the two graphs.

25. The directors of an exhibition published every four weeks the number of persons who paid for admission during the preceding four weeks. The numbers are given (roughly) in the following table. Exhibit them in a column-graph.

Convert this into a graph which will show the probable number of paying visitors in any given time. About how many people probably paid for admission during (i) the 6th, (ii) the 14th, (iii) the last week of the exhibition? In which week did the number first exceed 10,000, and in which week did it again fall below this number?

Month	1	2	3	4	5	6	7
Number admitted (in thousands)	22	36	42	30	22	15	8

D.

26. If the ends of a flexible chain are fastened to two pegs on the same horizontal level the chain hangs in a characteristic curve called the **catenary.** Hang up a chain (*e.g.* a watch-chain or neck-chain or a dog-chain with small links) against an upright blackboard or drawing-board. Fix the position of various points of the chain by measuring their distances above definite points on a base-line. Use these measurements to obtain a reduced reproduction of the catenary.

27. A steamer is travelling in an easterly direction across a bay. At a certain moment it is due north of a battery on shore and its distance (as determined by a range-finder) is four miles. The following table gives the distance of the steamer at subsequent times and its **bearing**, that is, the angle which the line to it from the battery makes with the north. Draw a diagram of its path across the bay.

Distance (in miles)	3·7	3·65	3·95	4·75	5·35	6·4	7·4	7·8
Bearing E. of N.	$14\frac{1}{2}°$	35°	$50\frac{1}{2}°$	$66\frac{1}{2}°$	74°	82°	$90\frac{1}{2}°$	$94\frac{1}{2}°$

28. Two lighthouses, A and B, are situated due north and south of one another and a mile apart. The following table gives the bearings of a ship taken simultaneously at A and B every ten minutes. Give a diagram of the path of the ship.

Bearings from A W. of N.	13°	$23\frac{1}{2}°$	$34\frac{1}{2}°$	$45\frac{1}{2}°$	54°	63°	$73\frac{1}{2}°$	$83\frac{1}{2}°$	90°
Bearings from B W. of N.	11°	$19\frac{1}{2}°$	29°	$38\frac{1}{2}°$	46°	54°	$62\frac{1}{2}°$	$72\frac{1}{2}°$	80°

29. A bicyclist is riding at constant speed along a road across a common. He whistles to his dog who is on the common at some distance from him. The dog constantly directs his motion towards his master and runs half as fast

again as the latter rides. Find, according to the method explained to you, the path of the dog.

30. A piece of mud on the rim of a moving cart traces out a curve called the **cycloid.** Trace a cycloid on paper by rolling a circle (*e.g.* a penny) along the edge of your ruler and marking a number of the positions occupied by a certain point on its circumference.

EXERCISE III.

THE WRITING OF FORMULÆ.

In order that your answers may be corrected easily, you should use the symbols placed in brackets after the statement of each problem. The first letter is always intended to be the symbol of the subject of the formula. The other letters are intended to be the symbols of the other phrases in the order in which they occur in the statement. The same symbols should be used in each of the different parts of the same question.

In many cases an arithmetical example of the kind of problem you are to deal with is first given. By considering how you would work this example, you can always find out the general rule for solving problems of that particular kind. In other cases, where you cannot see what is the rule for solving the problem, you should make up an arithmetical example for yourself and consider how you would solve it. You will in that way be able to find out the rule.

A.

1. Write down formulæ to find :—

(i) The weight of a bag containing 37 marbles, given the weight of the bag and the weight of a single marble (W, b, m);

(ii) The weight of a bag containing any given number of marbles (n);

(iii) The weight of a bottle of ink, given the weight of the bottle, the weight of a cubic inch of the ink, and the quantity in the bottle (W, b, i, V);

(iv) The length of shelf that would be occupied on a bookshelf by 8 volumes of a certain thickness, followed by 5 volumes of another given thickness (l, t_1, t_2);

(v) The length of shelf that will be occupied by a certain number of volumes $\frac{3}{8}$ inch thick, followed by a given number of volumes $\frac{7}{8}$ inch thick (l, n_1, n_2);

(vi) The length of shelf that would be occupied by two sets of volumes, given the number and the thickness of the volumes in each set;

(vii) The weight of a handful of new pence, shillings, and half-crowns, given the number and the weight of each kind of coin (W, n_1, n_2, n_3, p, s, h);

(viii) The value of the handful of coins expressed in pence (P).

2. (i) A boy goes into an office where his salary begins at £50 and increases by £5 every year. What will his salary be after a given number of years? (S, t).

(ii) Write down in symbols the general rule for finding the salary when the commencing salary, the annual increase, and the time of service are given (S, S_0, i, t).

3. (i) A motor-car begins to descend a hill at the rate of 14 miles an hour and increases its speed by 3 miles an hour every minute. How fast will it be moving after a given number of minutes? (s, t).

(ii) Write the rule so that it will apply to any initial speed and rate of increase (S, S_0, i, t).

4. (i) A man decides to spend £35 upon books for his library and buys them at the rate of £2 a month. What sum will be left at the end of a given number of months? (S, t).

(ii) Write the rule for solving all such problems, given the original sum of money and the monthly rate of expenditure (S, S_0, r).

5. (i) The brakes of a railway train are put on when it is going at 45 mls./hr. (miles per hour), and its speed now diminishes at the rate of 10 mls./hr. every minute until it stops. Write a formula for the speed of the train after a given time expressed in minutes (V, t).

(ii) Write this rule so that it will apply to any original speed and rate of decrease (V, V_0, r, t).

(iii) Write a formula for the speed of the train if, after its speed has been decreased at a certain rate for a given time, the brakes are taken off and the speed increases at a given rate for a given number of minutes (V, V_0, r_1, t_1, r_2, t_2).

6. (i) A boy sells a model flying machine and a number of white mice and spends most, but not all, of the proceeds in buying rabbits. Write a formula for the money left over (the "residue"), given the sum he receives for the flying machine, the prices of a mouse and a rabbit, and the numbers sold and bought (R, f, m, r, n_1, n_2).

(ii) Give a formula for the money he still requires if the cost of the rabbits is greater than the proceeds of his sale (R).

7. Write formulæ to find:—

(i) The area of the wall of a room containing a window, given the height and breadth of the wall and the height and breadth of the window (A, H, B, h, b);

(ii) The same when there are three windows of the same size;

(iii) The same when there is a given number of windows of the same size (n).

8. A bath is fed by two taps each supplying 1·7 gallons per minute, and is emptied by a waste pipe carrying away 2·1 gallons per minute. Write down formulæ for finding :—

(i) The number of gallons in the bath if to begin with it has 40 gallons in it and you turn on one tap for a given number of minutes (V, t) ;
(ii) The same, if you turn on both taps ;
(iii) The same, if you let the water run out with both taps off ;
(iv) Write down in symbols the rule for solving all problems like (i), (ii), and (iii), given the number of gallons in the bath to begin with, the amount supplied per minute by each tap, and the amount carried away per minute by the waste pipe (V, V_0, s, w).

9. Find formulæ by which to express any given term of the following series of numbers (T_n, n) :—

(i) 3, 5, 7, 9, . . .
(ii) 8, 13, 18, 23, . . .
(iii) 21, $24\frac{1}{2}$, 28, . . .
(iv) 1·4, 2·1, 2·8, 3·5, . . .
(v) $\frac{2}{3}$, $1\frac{1}{24}$, $1\frac{5}{12}$, . . .
(vi) 94, 88, 82, . . .
(vii) 18·2, 16·4, 14·6, . .

10. Calculate the 10th term of each of the foregoing series of numbers.

11. Supposing that the bath in No. 8 contained 120 gallons when full, how would you find :—

(i) The time in which it could be filled by one of the taps ;
(ii) The same, both taps being turned on ;
(iii) The same, both taps and the waste being turned on ?

Write formulæ expressing the rules for solving problems like (i), (ii), and (iii), given the number of gallons in the bath when full, the number of gallons delivered per minute by each tap, and the number carried away per minute by the waste pipe (t, V, s, w).

12. How would you answer questions (i), (ii), (iii) if the bath contained 40 gallons at the beginning?

Write three formulæ expressing the rules you would follow, using the same symbols as in the previous question.

13. Write down formulæ for finding:—

(i) The area of a rectangle, given the length and breadth;
(ii) The length of a rectangle, given the area and the breadth;
(iii) The breadth of a rectangle, given the area and the length;
(iv) The weight of a figure (e.g. the map of Great Britain) cut out from cardboard, given the area of the figure and the weight of a square centimetre (or square inch) of the cardboard (W, A, w);
(v) The area of a figure cut from cardboard, given its weight and the weight of unit area of the cardboard;
(vi) The weight per square centimetre of a sheet of cardboard, given the length, breadth, and weight of a rectangle cut from it;
(vii) The area of a triangle, given the base and the altitude (A, b, h);
(viii) The weight of a triangle when cut from cardboard, given the weight of a square centimetre (or square inch) of the cardboard;
(ix) The weight of a triangle cut from a brass plate weighing 3·4 grams/cm.^{2} (grams per square centimetre), given the base and the altitude;
(x) The altitude of a triangle cut from the same brass plate, given the weight and the base.

14. (i) Give the rule for finding the amount of air available for each person in a class-room, given the number of cubic feet of air and the number of persons (a, V, n);
(ii) The same, the number of persons being calculated by counting the number of desks and adding one for the master or mistress;
(iii) The same, the number of persons being calculated by counting the number of "dual" desks (i.e. desks that hold two) and adding one for the master or mistress;
(iv) The same, when the desks are single but three of them are unoccupied. (The teacher is present);
(v) The same when the desks are dual and three are unoccupied (teacher present).

15. (i) To find the weight of a single lead shot I propose to weigh 150 together in a beaker. What formula should I use to find the weight of a shot, given the weight of the beaker with the shot in it and of the beaker when empty? (s, w_1, w_2).
(ii) Write a formula for this problem for any given number of shot (n).

16. There are a certain number of villas in a road, all of the same size. Two roads lead out of this road on one side and three on the other. These roads all have the same width.

Given this width and the length of the road find a formula for the frontage of each villa (f, n, w, l).

17. In many towns all the houses on one side of a street have odd numbers while those on the other side have even numbers.

(i) Write down the rule for finding the number of houses on the "odd number" ide when you know the number of the last house on that side (n, N).
(ii) Give the corresponding rule for the "even number" side.
(iii) Give the rule for finding the number of houses between those bearing two given odd numbers (not counting the houses that have those numbers) (n, N_1, N_2).
(iv) Find whether the same rule holds good for the "even number" side.

18. (i) New-laid eggs are 2s. a dozen. Write down a formula for the cost in shillings of a given number of eggs (c, n).
(ii) Write a formula that will serve whatever be the price per dozen (c, n, p).

19. A business firm sets aside at Christmas a sum of money to be distributed equally to the three senior clerks in each of their departments. Write down formulæ for:—

(i) The amount of the bonus given to each department (b, A, n);
(ii) The amount received by each clerk (c).

20. Upon a piece of land of a given area (in square yards) a given number of blocks of workmen's dwellings have been erected. Each block contains 22 tenements. Write down formulæ for:—

(i) The number of square yards of the site per block (b, A, n);
(ii) The number of square yards of the site per tenement (t).

B.

21. The value of a piece of land in a London suburb has risen in ten years from £120 to £430. How would you calculate the rise in value of another similarly situated piece which was worth £240 ten years ago?

(i) Write down in symbols a rule for finding the increase of value in all such cases, given the former and the present value of one piece of land and the former value of the other (i, v_1, v_2, V_1).
(ii) Write down a formula for the present value of the second piece of land (V_2).

22. In another part of London the rent of a house has gone down from £85 to £70 during the last fifteen years. If the value of other houses has decreased in the same proportion, how would you find the decrease in the rent of a house now rented at £42?

(i) Write a formula for the decrease of the rent in such cases, given the former and the present rent of one house and the present rent of the other (d, r_1, r_2, R_2).
(ii) Write also a formula for the former rent of the second house (R_1).

23. A number of friends go on a holiday together, sharing equally the living expenses and the travelling expenses. Given the amount of each kind of expense and the number of the friends, there are two ways in which they could "settle up" at the end of the holiday.

(*a*) Each person could hand over to the treasurer first his share of the living expenses then his share of the travelling expenses.

(*b*) The treasurer could add together the expenses of both kinds, and each person could pay his own share of the whole.

Write down two formulæ to express these two ways of proceeding (l, t, n, S).

24. A man leaves his house property and his bank shares, each to be divided equally among his children. Given the value of the house property and the bank shares and the number of children, write, as in the last example, two formulæ for the value of the legacy received by each child (l, h, b, n).

25. A man left property of a certain value to be sold and divided equally among his nephews and nieces. A certain amount had to be paid to the Inland Revenue Office as Estate and Succession Duties. Given the value of the property, the amount of the duties, and the number of persons, write down two formulæ for the share ultimately received by each (S, p, d, n).

26. Two families go on a holiday together with the understanding that each person shall pay the same proportion of the expenses. If there were 5 in the one family and 3 in the other, what share of the total expenses would be borne by each family?

Write a formula for calculating the share of each family, given the number in each and the total expenses (S_1, S_2, n_1, n_2, E).

27. In the following examples the population of the town is supposed to be changed only by deaths and new births. Emigration and immigration are supposed to balance one another.

The number of births in a town is 86 per month and the number of deaths 64 per month. What is the easiest way of finding the increase of the population in (say) 7 months?

(i) Write a formula for the increase of the population of the town in a certain time, given the monthly number of births and deaths (i, t, b, d).

(ii) The population of a town is increasing. Find a formula for its value after a certain number of months, given the original population and the monthly number of births and deaths (P, P_0, t, b, d).

(iii) Write a formula to suit the case of a town whose population is decreasing.

28. Two motor-cars start from the same place and travel in different directions at speeds of 18 and 16 miles per hour respectively. How would you most easily calculate their distance apart after 4 hours?

(i) Write a formula for the distance apart after a given time of two cars which start from the same point and travel opposite ways at a given speed (d, t, s_1, s_2);

(ii) The same, the cars being already a certain distance apart to begin with (d_0);

(iii) The same, the cars starting from the same point but in the same direction;

(iv) The same, the cars going in the same direction and the faster being a given distance ahead of the slower at the beginning (d_0);

(v) The same, the slower car being a given distance ahead of the faster at the beginning.

29. Between two mile-stones on a hill-side the road gradually rises. Given the numbers on the mile-stones and the total rise in feet write formulæ to give:—

(i) The rise in feet per mile (r, m_1, m_2, R);

(ii) The rise in feet per foot;

(iii) The rise in feet per 100 feet.

30. Between two given dates a certain total number of inches of rain has been recorded at Kew Gardens. Given the dates and the total rainfall write down formulæ for:—

(i) The average rainfall per year (r, d_1, d_2, R) ;
(ii) The average rainfall per month.

31. A man owns a certain number of houses receiving from each the same rent and having to pay out of this rent the same amount of ground rent.

(i) What is his net income from the houses ? (I, n, r, g).
(ii) How much (in shillings) does he pay in income tax at 1s. 2d. in the pound ? (T).
(iii) He leaves the property to his children. What income does each receive ? (Let c represent no. of children.)
(iv) How many shillings income tax does each pay at 1s. 2d. in the pound ?

32. In the great Paris flood of 1910 the depth of water was observed to increase at one place from 1·8 metres to 3·2 metres in 12 hours. How would you find the amount of increase to be expected in (say) 16 hours ?

(i) Write a formula to find in all such cases the increase during a given time, given the first and second observed depths of the water and the interval of time during which the increase took place (i, t, d_1, d_2, T).
(ii) Write a formula for the total depth of the water after a given time (d).
(iii) Write a formula for the depth after a given time supposing that the depth of the water is decreasing instead of increasing (i.e. $d_1 > d_2$). Use the symbols of (i) and (ii).
(iv) Write a formula for the time the flood would take to disappear if it continued to subside at the same rate (T).

33. How many planks 10 inches wide would there be side by side across the floor of a room 16 feet wide ? The number could be determined either by reducing the 16 feet to inches or expressing the 10 inches in feet. What would the working be if you chose the latter way ?

(i) Find the number of 10 inch planks when the width of the floor is given in feet (n, w).
(ii) Find the number of 8 inch planks when the width of the floor is given in feet.

34. (i) How many books $\frac{3}{4}$ inch thick can be set side by side on a shelf of a given length in inches ? (n, l).
(ii) The same, the length being given in feet ?
(iii) The same, the books being $1\frac{1}{4}$ inch thick and the length given in inches ?
(iv) The same, the length being given in feet ?

35. How many planks 8 inches wide and 13 feet long would cover a floor whose area is 391 square feet? Write a formula for :—

(i) The number of planks 8 inches wide and of given length (in feet) that would cover a floor of a given area (in square feet) (n, l, A);
(ii) The same, the planks being 9 inches wide;
(iii) The same, the planks being 10 inches wide.

36. The subscribers to a library pay a subscription of 2s. 6d. a year and 2d. for every volume borrowed.

(i) Write a formula for the year's cost in pence to a person who borrows a certain number of books (C, n).
(ii) Write a formula for the year's cost in shillings.
(iii) Write a formula for the number of books borrowed, the cost being given in pence;
(iv) The same, the cost being given in shillings.

37. Men were admitted to a football match for 3d. and boys for 2d. There were 2036 men present and the receipts amounted to £32 9s. How would you find how many boys were present?

Write a formula for finding :—

(i) The number of boys when the number of men and the total receipts (in pounds) are given (b, m, T);
(ii) The number of men when the number of boys and the total receipts in pounds are given;
(iii) The total number of persons present in case (i) (n);
(iv) The total number present in case (ii).

38. At a cricket match 6d. is charged for admission to the ground and a further charge of 1s. for a seat in the grandstand. Given the number of persons in the grandstand and the total receipts (in pounds), find a formula :—

(i) For the number admitted but not to the grandstand (n_1, n_2, T);
(ii) For the total number admitted (n).
(iii) Write formulæ to calculate the same two numbers in any case, given (in shillings) the charge of admission to the ground and the extra charge for the grandstand (a, s).

39. (i) A bookshelf containing a certain number of volumes of the *Temple Shakespeare* is hanging from a nail on a wall. Write a formula for the pull on the nail, given the weight of the shelf and cord and the average weight of a volume (P, n, W, w).

(ii) Write a formula for the average weight of a volume, given the pull on the nail, the weight of the shelf and cord, and the number of volumes.

40. (i) A man has to walk to a place 24 miles away. How far will he be from his destination after a given number of hours if his rate of walking is $3\frac{1}{2}$ miles an hour? (d, t).

(ii) How many more hours will his journey take him? (T).

(iii) Write these rules so that they will apply to any given distance and rate of walking (d_0, r).

EXERCISE IV.

THE READING AND USE OF FORMULÆ.

A.

1. Each of the following formulæ tells you how to obtain the various terms of some series of numbers. Calculate the first four terms of each series and the 10th term.

(i) $T_n = 2n - 1$.

(ii) $T_n = 4n - 3$.

(iii) $T_n = 100 - 6n$.

(iv) $T_n = 4(3n - 2)$.

(v) $T_n = \frac{1}{3}(8n + 1)$.

(vi) $T_n = (n - 1)(n - 1)$.

(vii) $T_n = (2n - 1)(n + 3)$.

(viii) $T_n = n^3 - 2n + 7$.

(ix) $T_n = 2n(3n^2 - 1)$.

(x) $T_n = \dfrac{n + 4}{n + 1}$, or $T_n = (n + 4)/(n + 1)$.

(xi) $T_n = \dfrac{1}{2n - 1}$, or $T_n = 1/(2n - 1)$.

(xii) $T_n = \dfrac{4}{(2n - 1)(n + 3)}$, or $T_n = 4/(2n - 1)(n + 3)$.

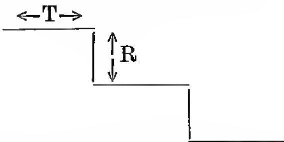
2. A young man has offers of three appointments. The salary in pounds (S) which he would earn after a given number of years' service (n) is given for each post by one of the following formulæ. In each case the salary ceases to rise after 20 years. Describe in words the various conditions offered, and calculate which will eventually give him the highest salary.

(i) $S = 120 + \frac{5}{2}n$.

(ii) $S = 90 + 4n$.

(iii) $S = 85 + \frac{15}{2}n$.

3. The "rise" in inches (R) in the steps of a staircase is connected with the "tread" (T) by the following formula (called "the French rule"). (i) Explain the rule in words; (ii) calculate the rise for a tread of (*a*) $9\frac{1}{2}$ inches, (*b*) $10\frac{3}{4}$ inches; (iii) find whether this rule has been followed in making the school staircase.



$$R = \tfrac{1}{2}(24 - T) \qquad 7 > R > 5\tfrac{1}{2}$$
$$12 > T > 9$$

4. The load that may be safely attached to an iron chain is given by the following formula:—

$$L = 7{\cdot}11d^2$$

$L \equiv$ safe load in tons; $d \equiv$ diameter of chain-iron in inches.

Find the greatest load that can be lifted by a chain in which the diameter of the iron is (i) half an inch, (ii) 1·2 inches.

5. The rule used by military engineers in blowing a breach in a wall by means of blasting powder is

$$p = 0{\cdot}1L^3$$

$p \equiv$ charge in lb.; $L \equiv$ half-thickness of wall in feet.

Explain the rule and calculate the charge necessary to make a breach in a wall (i) 3 feet thick, (ii) 14 feet thick.

Note.—The wall is bored half through.

6. The distance to which you can see from the top of a sea-cliff is given by the formula

$$d = 1{\cdot}22\sqrt{h}$$

$d \equiv$ distance in miles; $h \equiv$ height of cliff in feet.

Find how far you can see from a cliff (i) 25 feet high, (ii) 256 feet high, (iii) 400 feet high.

7. In the following formula $d \equiv$ the depth of the crown of a brickwork arch, $r \equiv$ the radius of the arch. Both are measured in feet. For a single arch $n = 0{\cdot}4$, for a series of arches $n = 0{\cdot}45$. Calculate the depth of the crown (i) of a single arch of radius 16 feet, (ii) of a series of arches of radius 9 feet.

$$d = n\sqrt{r}.$$

Note.—In this example n is called a "coefficient" or a "constant". What are the coefficients or constants in Nos. 4, 5, 6?

8. The velocity of the stream at the bottom of a river (v) is connected with the velocity at the surface (V) by the formula

$$v = (V + 1) - 2\sqrt{V}.$$

Calculate the velocity at the bottom when the velocity at the surface is (i) 4 ft./min.; (ii) 9 ft./min.; (iii) 121 ft./min.

9. Suppose you were asked to find by the formula of No. 6 the distance visible from a cliff 40 or 56 or 75 feet high. You could not do it easily because it is not easy to see the square roots of 40, 56, 75, etc. They must, no doubt, lie between the square roots of 36 and 49, 49 and 64, 64 and 81, respectively. It is possible, therefore, that they may be obtained by drawing a graph. Draw a graph as you are instructed and test it by choosing any number you please and seeing whether the graph gives the square root correctly.

How could you use the graph to find the square roots of numbers 100 times those which you have plotted?

Note.—This graph will be required in solving many of the following problems. It must therefore be drawn very accurately.

10. Find the distance visible from a cliff (i) 40, (ii) 56, (iii) 75, (iv) 200, (v) 770 feet high.

11. From a cliff I can just see the lights of a seaport 15 miles across the sea. Supposing that the lights are 20 feet above the sea, what is the height of my eye? (Use the graph of No. 9.)

12. Find the velocity at the bottom of a river when the velocity at the surface is (i) 30, (ii) 88 feet per minute.

13. Draw a graph to show how rapidly the charge of powder required to blast a wall increases as the thickness of the wall increases.

14. Use this graph to find the thickness of wall that requires a charge of (i) 20 lb., (ii) 35 lb.

How can you use this graph for finding cube roots?

15. Surveyors use the following rule to calculate roughly the heights of buildings, etc., when they have no instruments. Take on level ground a line AB of convenient length pointing towards the building. Set up at A a staff higher than yourself. Walk back from it until the line of sight through

the top of the staff meets the top of the building. Move the staff to B and repeat. Let H ≡ ht. of building, $h \equiv$ ht. of staff, E ≡ ht. of eye above ground, D ≡ distance AB, d_1 and $d_2 \equiv$ distances from A and B at the first and second observations respectively. Then :—

$$H = \frac{D(h - E)}{d_1 - d_2} + h.$$

To find the height of a church tower I set up a staff 12 feet high at two points distant 100 feet from one another. The line of sight cuts the top of staff and tower when I stand 11 feet from the staff at one end of the base, and when I stand 8 feet from it at the other end. My eye is 5 feet from the ground. Calculate the height of the tower.

Calculate the height of a tree given D = 50 feet, h = 10 feet, E = 4 feet 9 inches, d_1 = 9 feet, d_2 = 6 feet 6 inches.

16. Find the height of the school or a neighbouring building, etc., by the method of No. 15. Compare your results with those obtained by others in the class, and, if possible, with the known height of the object.

B.

17. A beam is fixed in masonry at one end and sticks out horizontally into a room. It is loaded at the free end. The following formula gives the weight it will just bear without breaking :—

$$W = \frac{kbd^2}{l}$$

W ≡ breaking weight in cwt.; $b \equiv$ breadth of beam, and $l \equiv$ length of beam, $d \equiv$ depth or thickness of beam—all in inches. k is a coefficient. For wrought iron k = 68, for cast iron k = 46, for English oak k = 15.

(i) Find the greatest weight that can be hung on the end of a wrought-iron bar 10 feet long and 2 inches square;

(ii) The same, the bar being of cast iron.

(iii) What weight will break an oak beam 12 feet long, 3 inches wide, and 4 inches deep?

18. The quantity of water raised per minute by a pumping engine (e.g. from the hold of a ship or from a well) is given by the formula :—

$$g = 0{\cdot}034\, nld^2$$

$g \equiv$ no. of gallons per minute; $n \equiv$ no. of strokes per minute; $l \equiv$ length of stroke in feet; $d \equiv$ diameter of pump in inches.

(i) Find how much water can be raised by a pumping engine making 20 strokes a minute, 2 feet 6 inches in length, the diameter of the pump barrel being 22 inches;
(ii) The same, when $n = 15$, $l = 3$ feet, $d = 18$ inches.

19. In using a rope to lift heavy weights, etc., it is important to know (*a*) the working load (i.e. the weight that the rope can be used to lift constantly) and (*b*) the breaking load. These are given by the formulæ:—

$$(a)\ \mathrm{L} = k_1\mathrm{C}^2; \quad (b)\ \mathrm{B} = k_2\mathrm{C}^2$$

$\mathrm{L} \equiv$ working load in tons; $\mathrm{B} \equiv$ breaking load in tons; $\mathrm{C} \equiv$ circumference of rope in inches; k_1 and k_2 are "constants" depending on the material.

	k_1	k_2
Common hemp . . .	0·032	0·18
Best hemp . . .	0·100	0·60
Iron-wire rope . . .	0·290	1·80
Steel-wire rope . . .	0·450	2·80

Calculate the working load (i) for a common hemp rope 4 inches in circumference; (ii) for a steel-wire rope of the same circumference. Calculate (iii) the breaking load for an iron-wire rope 2 inches in circumference. (iv) Could a steel rope 1 inch in circumference bear safely the working load of an iron-wire rope of 3 inches in circumference?

20. Draw, on a single sheet of squared paper and with the same axes, graphs showing the working load of iron-wire and steel-wire ropes for different circumferences up to 4 inches. Be careful to distinguish the curves by labels.

21. Answer the following questions from the graphs:—

(i) What are the working loads of an iron-wire rope and of a steel-wire rope of $2\frac{1}{2}$ inches circumference?
(ii) What should be the circumference of a steel-wire rope for a crane which has to lift $4\frac{1}{2}$ tons? If the builder had to use an iron-wire rope what should be its circumference?
Measure, if you can, the circumference of some thick hemp or a wire rope and calculate its working and breaking loads.

22. When a ship springs a leak the quantity of water (in tons per hour) is given by the formula:—

$$\mathrm{W} = \mathrm{A}\sqrt{20d}$$

A ≡ area of hole in square inches; d ≡ depth of hole in feet.

(i) A rivet with an area of $\frac{1}{2}$ square inch has fallen out of a ship's bottom 20 feet below the water-line. How many tons of water must be pumped out of the ship per hour so that it may not accumulate?

(ii) There is a hole 4 square inches in section in the side of a ship 4 feet 3 inches below the water-line. How many tons an hour will leak in? (Answer to the nearest ton.)

23. If you look at a crane you will see a large hook at the end of the rope. The weakest part of the hook is the "shank," i.e. the upright part that passes through the iron attachment to the rope. By the following formula the engineer calculates the thickness of shank necessary if a given load is to be lifted. D ≡ diameter of shank in inches; W ≡ weight to be lifted in tons.

$$D = \sqrt{0{\cdot}45W} + 0{\cdot}2$$

What is the size of shank needed for dealing with the following loads?—(i) 20 tons, (ii) 10 tons, (iii) 16 tons. (Use the graph of No. 9.)

24. Piles are driven into the ground by a "pile-driver," a machine that lifts a heavy weight called a "ram" and drops it on to the pile. If a house or other load is to be laid upon the piles we must know how much each pile can support. The formula is:—

$$L = \frac{Wh}{d(W + P)}$$

L ≡ greatest load in tons pile will bear; W ≡ weight of ram in cwt.; h ≡ height in feet from which ram falls; d ≡ distance in inches that the pile was driven in by the last blow; P ≡ weight of pile in cwt.

A ram weighing 6 cwt. falling a height of 4 feet drove a pile weighing 15 cwt. $1\frac{1}{2}$ inches into the ground. What is the greatest load the pile will safely bear?

(*Note.*—It is usual actually to load the pile with not more than about $\frac{1}{5}$ of this load. Compare the working and greatest load of ropes in No. 19.)

25. Fig. 3 represents one span of a telephone wire supported by standards at A and B. An engineer uses the symbols l for the bare length of the span between A and B,

L for the actual length of wire in the span, d for the dip or sag of the wire in the middle, all these measurements being made in feet. He also uses the symbol s for the pull with which the wire is stretched over the standards and w for the weight of every foot of the wire, both being measured in pounds. Explain the meaning and use of the following formulæ taken from his pocket book :—

FIG. 3.

$$\text{(i)} \quad \mathrm{d} = \frac{\mathrm{l^2w}}{\mathrm{8s}}$$

$$\text{(ii)} \quad \mathrm{s} = \frac{\mathrm{l^2w}}{\mathrm{8d}}$$

$$\text{(iii)} \quad \mathrm{L} = \mathrm{l} + \frac{\mathrm{8d^2}}{\mathrm{3l}}$$

$$\text{(iv)} \quad \mathrm{d} = \sqrt{\frac{\mathrm{3l(L - l)}}{8}}$$

26. An iron telephone wire weighs 0·072 lb./ft., and is stretched with a pull equal to the weight of 270 lb. between standards 100 feet apart. Calculate the dip of the wire in inches.

27. The dip of the same wire stretched between standards 120 feet apart is observed to be 6 inches. Calculate the stretching force.

28. What is the actual length of telephone wire in the loop between two standards 150 feet apart when the dip is 18 inches?

29. The size of the barrel of the winding engine used to raise and lower the "skips" in a coal-pit is given by the formula :—

$$D = \frac{12p - 3{\cdot}15n^2t}{37{\cdot}7n}$$

$D \equiv$ diameter of winding barrel in feet; $p \equiv$ depth of pit in feet; $n \equiv$ no. of revolutions of engine per min.; $t \equiv$ thickness of rope in inches.

Calculate the size of the winding barrel for a pit-shaft 800 feet deep, the rope being 3 inches thick, and the number of revolutions of the engine 30 per minute.

C.

Note.—The most important parts of an ordinary steam engine are (a) the cylinder C (fig. 4) into which the steam

is admitted from the boiler by a valve; (*b*) the piston P which is driven up the cylinder by the steam; (*c*) the piston-rod R_1; (*d*) the crank-shaft S which turns the wheel round; (*e*) the connecting-rod R_2; (*f*) the crank-pin *p* which fastens the crank-shaft and the connecting-rod together.

When the steam enters the cylinder it presses against the piston with a certain force. Imagine the cylinder held upside down and weights to be packed on it till they press the piston down exactly as hard as the steam did. Then the weight lying on each square inch of the piston is called the steam-pressure. As the piston moves out the steam-pressure will not remain steady, but we can imagine a steady steam-pressure which would produce on the whole the same effect. This is called the "mean" steam-pressure.

James Watt found that the work performed by his engines

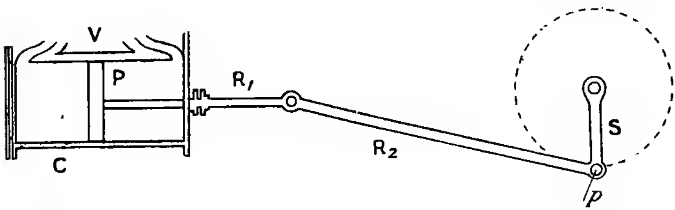


FIG. 4.

could, as a rule, be accounted for by supposing a steady steam-pressure of 7 lb. per square inch. He always assumed this to be the pressure, therefore, in calculating the "horse-power" of his engine. In these days much greater steam-pressures are used, but makers in describing their engines sometimes still follow Watt's plan. The horse-power calculated according to Watt's rule is called the "nominal horse-power" of the engine (N.H.P.). The actual horse-power is called the "indicated horse-power" (I.H.P.).

The following symbols are used in Nos. 30-33:—

$P \equiv$ mean steam-pressure in lbs. per square inch.

$D \equiv$ diameter in inches of cylinder.

$l \equiv$ the length of stroke of the piston in inches, i.e. the distance it is driven forward at each revolution of the wheel.

$n \equiv$ number of revolutions of the wheel per minute.

$H \equiv$ the indicated horse-power.

$N \equiv$ the nominal horse-power.

30. The tractive force or pull of a locomotive engine is given by the formula :—

$$T = \frac{PD^2 l}{w}$$

$T \equiv$ tractive force in lbs. ; $w \equiv$ diameter of the engine's driving wheel in inches.

(i) Find the tractive force of an engine with a driving wheel 5 feet high, the cylinders being 2 feet in diameter, the length of stroke 3 feet 6 inches and the mean steam-pressure 50 lb. ;

(ii) The same, given that $P = 55$, $D = 27$, $l = 45$, $w = 66$.

31. A certain amount of the pressure of the steam is necessarily wasted in overcoming the friction of the piston against the cylinder. The amount of this pressure is given by the formula :—

$$P = \frac{18}{\sqrt{D}}$$

Draw a graph showing the loss of steam-pressure to be expected in cylinders from 1 foot to 4 feet in diameter. (Take $\frac{1}{10}$ inch to represent an inch in the diameter of the cylinder.) What is the loss with cylinders of the following diameters: (i) 16 inches, (ii) 23 inches, (iii) 40 inches? What is the smallest cylinder for which the loss of pressure will be not more than (iv) 4 lb., (v) 3 lb.?

32. The old Admiralty formula for calculating the N.H.P. of a paddle engine was

$$N = \frac{nlD^2}{3000}$$

(i) Calculate the N.H.P. of an engine when the diameter of cylinder is 54 inches, stroke 36 inches, and revolutions per minute 30 ;

(ii) The same, substituting $D = 48$ inches, $l = 40$ inches, $n = 35$.

33. The indicated horse-power may be calculated by the following formula. Use it to find the I.H.P. of a marine engine in which (i) the diameter of the cylinder is 5 feet 10 inches, the length of stroke 4 feet, the mean steam-pressure 30 lb., and the number of revolutions 15; (ii) $D = 5$ feet, $l = 5$ feet 3 inches, $n = 20$, $P = 20$ lb.

$$H = \frac{nlD^2P}{21{,}000}$$

EXERCISE V.

FACTORIZATION (I).

1. Calculate the total floor space of two rooms arranged as in fig. 5 (p. 34) when the dimensions are as follows:—

	AB	BC	AD
(i)	24 ft.	16 ft.	13 ft.
(ii)	119 ft.	81 ft.	64 ft.
(iii)	27·3 ft.	32·7 ft.	16·5 ft.
(iv)	21 ft. 4 in.	18 ft. 8 in.	14 ft. 6 in.

2. Calculate the area of a room shaped like fig. 6 when the dimensions are as follows:—

	AB	AF	ED	DC
(i)	40 ft.	20 ft.	35 ft.	20 ft.
(ii)	25 ft.	17 ft.	15 ft.	17 ft.
(iii)	18·6 ft.	13·2 ft.	11·4 ft.	13·2 ft.
(iv)	21 ft.	17 ft.	21 ft.	13 ft.
(v)	26 ft.	22 ft.	26 ft.	18 ft.
(vi)	36 ft. 6 in.	21 ft. 7 in.	36 ft. 6 in.	28 ft. 5 in.

3. Fig. 7 represents the plan of a picture gallery. Find its area when the dimensions are as follows:—

	AB	AH	DE	CD
(i)	47 ft.	15 ft.	53 ft.	15 ft.
(ii)	52 ft.	23 ft.	52 ft.	27 ft.
(iii)	44 ft.	16 ft.	28 ft.	32 ft.[1]
(iv)	57 ft.	14 ft.	$26\frac{1}{2}$ ft.	28 ft.
(v)	120 ft.	37 ft.	60 ft.	26 ft.
(vi)	88 ft.	17 ft.	44 ft.	16 ft.

4. Write down a formula for the total floor space in the rooms represented in fig. 8. Express the formula in the form most suitable for calculation. (Use a, b, c as symbols for the lengths of the rooms, d for their common width.)

[1] Imagine the rectangle CE to be divided into two rectangles 16 feet wide.

5. Obtain formulæ for calculating easily the area of a T-shaped room like fig. 7 :—

(i) When AB (a) and DE (b) are unequal but AH and CD are equal (c) ;
(ii) When AB and DE are equal (a) but AH (b) and CD (c) are unequal ;
(iii) When AB (a) and DE (b) are unequal and CD is double AH (c) ;
(iv) When AB (a) and DE (b) are unequal and AH is three times CD (c) ;
(v) When AB is double DE (a) and AH (b) and CD (c) are unequal ;
(vi) When DE is four times AB (a) and AH (b) and CD (c) are unequal.

6. The wall on one side of a passage is 36 feet long and 14 feet high. It is faced with bricks up to a height of 4 feet, but above this height is covered with paint. Calculate :—

(i) The area of the painted surface in square feet ;
(ii) The area of the same in square yards ;
(iii) The cost of painting it at 2d. a square yard.

7. Write down formulæ (i) for the area (measured in square feet) of the painted surface of the last question ; (ii) for the cost of painting it. Let $a \equiv$ the length in feet, $b \equiv$ the height in feet of the passage, $c \equiv$ the height in feet of the brickwork, $p \equiv$ the cost in pence of the paint per square yard, $C \equiv$ the total cost in shillings. The formulæ are to be expressed in the form most suitable for calculation.

Note.—Instead of writing " Let a be the symbol for the length in feet of the passage," " let p be the symbol for the cost in pence of the paint per square yard," it is often convenient (because shorter) to write " Let the passage be a feet long," " let the paint cost p pence per square yard". Upon this plan the foregoing question could be expressed as follows : " Find the total cost in shillings (C) of painting a passage a feet long and b feet high, but faced with brickwork up to a height of c feet, the cost of the painting being p pence per square yard ".

8. Across the upper part of the wall in No. 6 there runs a board d feet wide to carry hat-pegs. Write down formulæ for calculating in the easiest way (i) the area of the painted surface ; (ii) the cost of painting it. (Use the same symbols as in No. 7.)

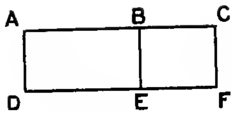


Fig. 5.

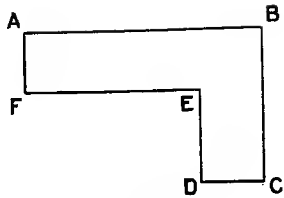


Fig. 6.

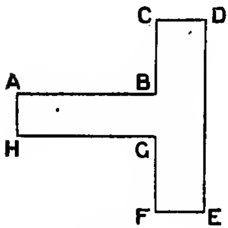


Fig. 7.

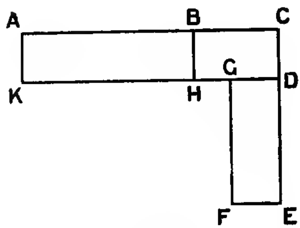


Fig. 8.

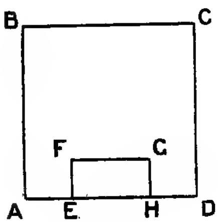


Fig. 9.

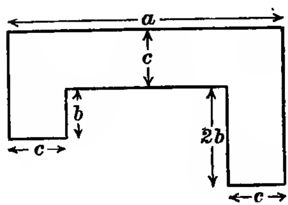


Fig. 10.

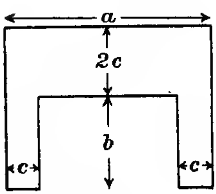


Fig. 11.

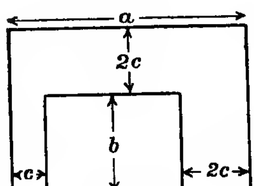


Fig. 12.

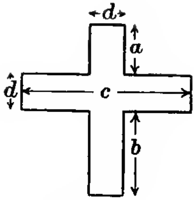


FIG. 13.

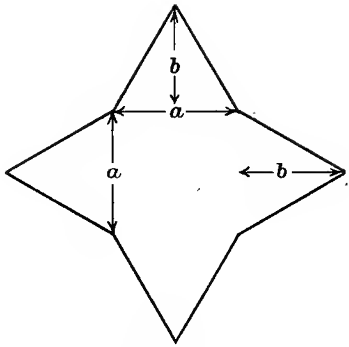


FIG. 14.

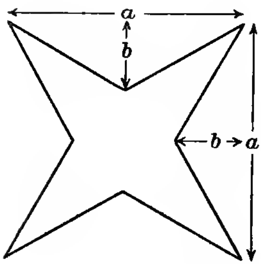


FIG. 15.

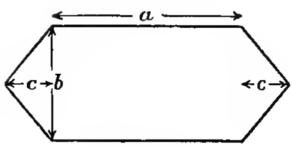


FIG. 16.

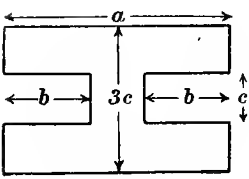


FIG. 17.

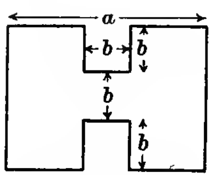


FIG. 18.

9. A strip of carpet c feet wide runs along the middle of a passage a feet long and b feet wide. Write down formulæ for calculating in the easiest way (i) the area of the uncovered part of the floor; (ii) the total cost in pence (C) of polishing it at a cost of p pence per square foot; (iii) the total cost in shillings (C) of polishing it at a cost of p pence per square yard.

10. Across a room 97 feet long and 35 feet wide (AF in fig. 5) a partition is thrown so as to cut off a room 47 feet long (AE). Find the area of the room (BF) on the other side of the partition.

11. Across a room a feet long and c feet wide a partition is thrown so as to cut off a room b feet long. Find[1] the area of the room on the other side of the partition in a form suitable for easy calculation.

12. Find the cost in shillings of covering with linoleum at l shillings per square yard the room whose area was calculated in the last question.

13. Fig. 9 is the plan of a hall AC containing a platform EG. Seats are to be placed to the right and left of the platform as well as in front of it. Calculate the area available for seats when the dimensions are as follows:—

	AB	AD	EF	EH
(i)	109 ft.	40 ft.	18 ft.	20 ft.
(ii)	107 ft.	32 ft.	14 ft.	16 ft.
(iii)	89 ft.	57 ft.	27 ft.	19 ft.
(iv)	117 ft.	46 ft.	23 ft.	34 ft.
(v)	99 ft.	55 ft.	15 ft.	33 ft.

14. Write down formulæ for the available floor space in the hall represented by fig. 9 in the following circumstances. (The formula is in each case to be in the form most suitable for substitution):—

(i) AB (a) and EF (b) are unequal while AD is double EH (c);
(ii) AB is double EH (a) while AD (b) and EF (c) are unequal;
(iii) AB (a) and EH (b) are unequal while AD is three times EF (c);
(iv) AB (a) and EF (c) are unequal while EH (b) is two-thirds of AD;
(v) AB (a) and EF (c) are unequal while EH is three-quarters of AD (b).

[1] That is, write down a formula for finding the area of, etc. The expression " find the area, etc.," is used for brevity.

15. Write down formulæ for the areas exhibited in figs. 10-18. In each case throw the formula into the shape in which it is most convenient for use in calculating the area.

Note.—If you cannot easily find the formula it is often a good plan to copy the figure on a piece of paper or thin cardboard, and to seek a way of cutting it up and re-arranging its parts so as to form a simple area.

16. Calculate by short methods the value of each of the following numerical expressions:—

(i) $17{\cdot}4 \times 8{\cdot}6 - 7{\cdot}4 \times 8{\cdot}6$.
(ii) $17{\cdot}4 \times 8{\cdot}6 - 14{\cdot}8 \times 4{\cdot}3$.
(iii) $24 \times 13 + 9 \times 13 - 3 \times 13$.
(iv) $24 \times 13 + 9 \times 26 - 4 \times 39$.
(v) $12 \times 14 + 4 \times 28$.
(vi) $21 \times 16 - 35 \times 8$.
(vii) $38 \times 55 - 57 \times 33$.
(viii) $14 \times 15 \times 11 - 22 \times 7 \times 5$.
(ix) $\dfrac{13 \times 18 + 26 \times 12}{7 \times 23 - 14 \times 5}$.
(x) $\dfrac{9 \times 34 - 12 \times 17}{39 \times 7 + 21 \times 21}$.

17. Find identities by the aid of which formulæ containing the following expressions can be made more suitable for calculation:—

(i) $a^2b + a^2c$. (ii) $pqr - q^2$. (iii) $pq^2 - q^2r$.
(iv) $al^2m + blm^2$. (v) $a^3 - pa^2$. (vi) $pa^3 + q^2a^2$.
(vii) $abc + bcd - cde$. (viii) $2ab + 3ac$. (ix) $\frac{3}{2}p^2q - 4pq^2$.
(x) $a^2b^2 - 2ab$. (xi) $ap^2q - pq$. (xii) $ar^3 - 2br^2 + r$.

18. Reduce the following formulæ to the forms most convenient for substitution:—

(i) $A = 4l^2 - 5ld$.
(ii) $M = \frac{1}{2}a^2p + \frac{1}{3}apq$.
(iii) $V = \frac{4}{3}\pi r^3 + \frac{1}{3}\pi r^2h$.
(iv) $P = a^2bc - ab^2c + abc^2$.
(v) $Q = 2a^3b + 3ab^3 - ab$.
(vi) $T = ap^2q - 2bpq^2 + 3q^3$.
(vii) $W = 2\pi r^2w - 4raw$.
(viii) $B = 12\,m^2n + 9\,mn^2 - 3mn$.
(ix) $C = \frac{1}{3}(6a^3 - 15a^2 + 3a)$.
(x) $V = 3\pi a^2c + 4\pi abc$.

Write down formulæ in the most convenient form for the calculation of the following quantities. (The linear measurements

indicated in the figures may all be supposed to be made in centimetres) :—

19. The weight (W) of a brass plate cut to the shape of fig. 14 (the weight of a square centimetre being w grams).

20. The same for a plate cut to the shape of fig. 18.

EXERCISE VI.

FACTORIZATION (II).

1. Fig. 19 is the plan of a courtyard. AC is a square and FD if completed by producing the lines AF and CD would be also a square. Find the area of the courtyard when the dimensions are as follows:—

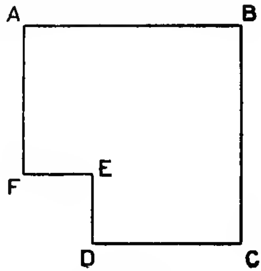


FIG. 19.

	AB	FE
(i)	129 ft.	29 ft.
(ii)	78 ft.	18 ft.
(iii)	146 ft.	54 ft.
(iv)	57·4 ft.	32·6 ft.

2. Fig. 20 (p. 41) represents a square metal plate with a square hole in the middle. Find the area of the plate when the dimensions are as follows:—

	AB	CD
(i)	14·5 cms.	5·5 cms.
(ii)	$6\frac{3}{4}$ in.	$1\frac{1}{4}$ in.
(iii)	$7\frac{5}{8}$ in.	$2\frac{3}{8}$ in.
(iv)	16·3 in.	4·3 in.

(Imagine the hole to be moved from the middle to the corner of the plate.)

3. Fig. 20 may also be supposed to represent the cross-section of a hollow metal bar. Find its volume when AB = 6·8 cms., CD = 3·2 cms., and the bar is 20 cms. long.

4. Write down formulæ for calculating quickly—

(i) The volume (V);

(ii) The weight (W) of the hollow bar of No. 3, using the symbols a and b for the length of AB and CD, l for the length of the bar, and w for the weight of 1 c.cm. of the metal.

5. Find the most useful formula for the weight w of a cubic centimetre of the material of the bar, given its total weight and its dimensions.

Before trying the next example answer the following questions :—

(i) How can 4 equal squares whose sides measure 3 inches be arranged so as to make a single square ?
(ii) What will be the length of the side of this square ?
(iii) Answer similar questions with regard to sets of 9, 16, and 25 equal squares ;
(iv) By what symbols will you represent the side of a square made up of 4 equal squares the length of whose sides is represented by b ?
(v) Answer the same question with regard to squares made up of 9, 16, and 25 of the smaller squares.

6. Fig. 21 represents a square metal plate measuring a cms. each way, in which four square notches have been cut, each measuring b cms. each way. Find in a form suitable for calculation the area of metal left.

7. Do the same for the plate represented by fig. 22.

8. Do the same for the plate represented by fig. 23.

9. Give convenient formulæ for calculating :—

(i) The weight (W) of the plate represented by fig. 22, given that the material weighs w grams/cm.^{2} ;
(ii) The weight (W) of a bar l cms. long whose cross-section is represented by fig. 21, given that the material weighs w grams/cm.^{3} ;
(iii) The weight (w) of a square centimetre of the plate from which fig. 23 is supposed to be cut, given that the whole plate weighs W grams.

10. Give the best formula for calculating the cost in shillings (c) per cubic foot of a rod whose cross-section is represented by fig. 23, given the dimensions in feet and the total cost (C) in pounds.

11. Write down the identities which you would use in simplifying for calculation formulæ that contain the following expressions. Show that the identities are always true no matter what quantities are or numbers the symbols refer to :—

(i) $p^2 - q^2$.	(ii) $4a^2 - b^2$.	(iii) $m^2 - 9n^2$.
(iv) $36a^2 - 25b^2$.	(v) $p^2a^2 - b^2$.	(vi) $u^2 - v^2t^2$.
(vii) $p^2u^2 - r^2t^2$.	(viii) $a^2b^2 - 16c^2$.	(ix) $a^2 - 16$.
(x) $81 - b^2$.	(xi) $p^2q^2 - 25$.	(xii) $1 - m^2n^2$.

Note.—Expressions such as $ac + bc$ and $a^2 - b^2$ can, as you have seen, be replaced for the purposes of easy calculation by expressions such as $(a + b)c$ and $(a + b)(a - b)$. These

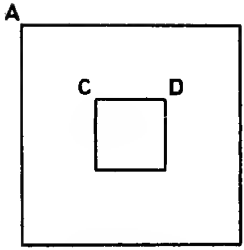


Fig. 20.

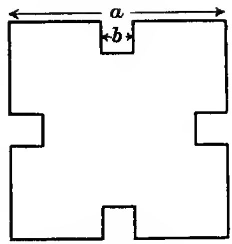


Fig. 21.

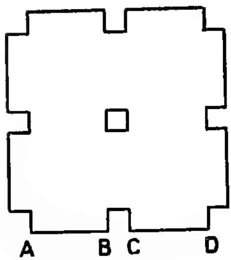


Fig. 22.

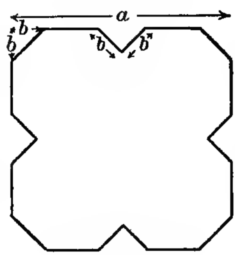


Fig. 23.

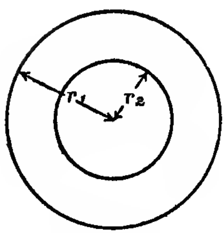


Fig. 24.

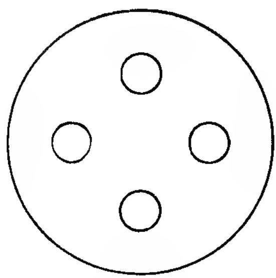


Fig. 25.

expressions describe **factors** which when multiplied together will give the result of the calculation. For this reason the task of discovering them is often called **factorizing** the original expressions.

12. Factorize the following expressions. (You need not prove that the identities are always true) :—

(i) $pa^2 - pb^2$. (ii) $4ap^2 - 9aq^2$. (iii) $\pi r_1{}^2 - \pi r^2{}_2$.
(iv) $a^3 - ab^2$. (v) $p^2t - t^3$. (vi) $4p^2t - 25t^3$.
(vii) $12a^2 - 27b^2$. (viii) $a^2b - 36b^3$. (ix) $8a^2 - 50$.
(x) $4p^3 - 9p$. (xi) $a^2b^2 - c^2$. (xii) $9a^3b^2 - a$.

13. Factorize the following expressions :—

(i) $(a^2 - b^2) + 3(a + b)$. (ii) $(a^2 - b^2) + 7(a - b)$.
(iii) $(a^2 - b^2) - 4(a + b)$. (iv) $(a^2 - b^2) - p(a - b)$.
(v) $(a^2 - b^2) + (a + b)$. (vi) $(a^2 - b^2) + (a - b)$.
(vii) $2(a^2 - b^2) + 6(a - b)$.. (viii) $2(a^2 - b^2) - 3(a + b)$.
(ix) $a(p^2 - q^2) + b(p - q)$. (x) $a(p^2 - q^2) - b(p + q)$.

14. Factorize the following expressions :—

(i) $(a + b)^2 - c^2$. (ii) $(p - q)^2 - r^2$.
(iii) $(a + b)^2 - 4$. (iv) $(p - q)^2 - 9$.
(v) $(p + q)^2 - 9r^2$. (vi) $(a - b)^2 - 16c^2$.
(vii) $4(u + v)^2 - w^2$. (viii) $9(u - v)^2 - 4w^2$.
(ix) $(a + b)^2 - a^2$. (x) $(a + b)^2 - b^2$.
(xi) $(a - b)^2 - b^2$. (xii) $(a - b)^2 - a^2$.
(xiii) $4(a + b)^2 - b^2$. (xiv) $25(a - b)^2 - a^2$.
(xv) $(a + b)^2 - 4b^2$. (xvi) $(a - b)^2 - 9b^2$.
(xvii) $16(p + q)^2 - 36p^2$. (xviii) $25(p - q)^2 - 9q^2$.

Note.—Before doing the next set of examples answer the following questions :—

(i) If from a group of 12 things I take all of them except 5 how many are left ?
(ii) If from a group of 29 things I take all except 13 (i.e. $29 - 13$) how many are left ?
(iii) What is left when I take $32 - 17$ from 32 ?
(iv) Complete the identity $a - (a - b) =$;
(v) What is the value of $48 + (48 - 7)$?
(vi) Complete the identity $a + (a - b) =$.

15. Factorize the following expressions :—

(i) $a^2 - (a - b)^2$. (ii) $p^2 - (p - q)^2$. (iii) $r^2 - (r - 3)^2$.
(iv) $p^2 - (p - 6{\cdot}7)^2$. (v) $ap^2 - a(p - q)^2$. (vi) $\pi r^2 - \pi(r - w)^2$.
(vii) $a^2 - (a - 3b)^2$. (viii) $s^2 - (s - rt)^2$. (ix) $a^2 - (a - pb)^2$.
(x) $p^2 - (p - aq)^2$.

Note.—In the following examples the formulæ are always to be thrown into the form most suitable for substitution.

16. Find the area of a figure shaped like fig. 19, given that FE $= a$ cms. and DC $= b$ cms.

17. Find the area of the same figure, given that DC $= a$ cms. and DE $= b$ cms.

18. Find the area of fig. 22, given that AB $= a$ inches and BC $= b$ inches.

19. Find the area of the same figure, representing the length of AB by p and the length of BC by q.

20. Find the area of fig. 22; representing the length of AD by p and the length of BC by q.

21. Find the area of the same figure, if $a \equiv$ the length of BC and $b \equiv$ the length of AB.

22. Find the area of figure 19, given that AB $= a$ cms. and DC $= b$ cms.

23. Calculate the area of the ring represented in fig. 24, given that $r_1 = 13{\cdot}9$ cms. and $r_2 = 11{\cdot}1$ cms. (Take $\pi = 22/7$.)

24. Write down formulæ for calculating (i) the area of a ring-shaped surface whose external and internal radii are r_1 cms. and r_2 cms. respectively; (ii) the weight (W) of such a ring cut out of a brass plate weighing w grams/cm.^{2}; (iii) the volume of l cms. length of a pipe whose external and internal radii are r_1 and r_2 cms.; (iv) the weight (W) of such a pipe, given that the material weighs w grams/cm.^{3}; (v) the weight (w) of a cubic centimetre of the material of such a pipe given the weight (W) of a l cms. length of it.

25. Fig. 25 represents a circular disc pierced by four equal circular holes. Write down a formula for the surface left, representing the radius of the disc by a and the radius of a hole by b.

EXERCISE VII.

SQUARE ROOT.

1. Find correctly to two decimal places (if necessary) the length of the side of a square whose area is (i) 23·04 square yards, (ii) 62·41 square inches, (iii) 94·09 square cms., (iv) 72 square feet, (v) 3·26 square miles, (vi) 28·7 square feet, (vii) 81·263 square yards, (viii) 1·024 square miles.

2. Find the square root of each of the following numbers (the roots are to be given correctly to three significant figures): (i) 348; (ii) 1562·8; (iii) 41616; (iv) ·043; (v) ·000294.

3. The total area of the British Isles is 121,377 square miles. If the land could all be arranged in the form of a square what would be the length of its side?

4. The area of England and Wales is 58,324 square miles. Calculate the length of the side of a square having this area.

5. Using the results of the last two examples construct on squared paper a diagram (arranged like fig. 20, p. 41) showing the proportion of the area of England and Wales to the area of the whole of the British Isles.

6. The total population of the British Isles in 1901 was 41,609,091, that of England and Wales also was 32,527,843. Construct a diagram, like the one of No. 5, showing what part of the total population lies within the boundaries of England and Wales.

7. Use the formula $d = 1{\cdot}22\sqrt{h}$ (see Ex. IV, No. 6) to calculate the greatest distance you could see across the water from a cliff 527 feet high, allowing 5 feet more for the height of the eye above the ground. Compare this result with that obtained graphically.

8. Make a similar calculation for the distance visible from a mountain 5320 feet high.

9. The Peak of Teneriff is 12,180 feet high. What is the greatest distance from which it could be seen by an observer at the mast-head of a ship 54 feet above the sea?

10. The velocity (V) of the water at the surface of a river is 18·5 feet per minute. Calculate the velocity at the bottom (v) by the formula :—

$$v = (V + 1) - 2\sqrt{V}$$

11. The top of the inner dome of St. Paul's Cathedral is so high above the floor of the nave that a small but heavy weight could swing beneath it at the end of a wire 285 feet long. Find how long each swing would take using the formula :—

$$t = 1{\cdot}11\sqrt{l}$$

in which $t \equiv$ time of a single complete swing in seconds, $l \equiv$ length of wire in feet.

12. How would you calculate the radius of a circle if you knew its area? Write down a formula.

13. Use the formula of the last question to calculate the radii of the circles whose areas are (i) 126·2 square inches, (ii) 1·82 square feet.

14. Calculate the radii of the two circles whose areas would be equal to those of the whole British Isles and England and Wales respectively (see Nos. 3 and 4). Use these circles to construct a diagram on the same principle as that of No. 5.

15. Give a formula for calculating the radius of a cylinder when you know its volume, and its height.

16. The volume of a cylinder 4·8 cms. high is 2·67 c.cms. What is its radius?

17. To find the thickness of some copper wire I immerse a piece of it 10 metres long in water in a measuring cylinder. The water rises 4·5 c.cms. Calculate the thickness of the wire.

EXERCISE VIII.

"SURDS."

Note.—It may be assumed in the following examples that $\sqrt{2} = 1{\cdot}41$, $\sqrt{3} = 1{\cdot}73$, $\sqrt{5} = 2{\cdot}24$, $\sqrt{7} = 2{\cdot}65$, $\sqrt{11} = 3{\cdot}32$, $\sqrt{13} = 3{\cdot}6$. What do these statements mean?

1. Find to two decimal places the value of the following:— $\sqrt{8}$, $\sqrt{12}$, $\sqrt{6}$, $\sqrt{39}$, $\sqrt{99}$, $\sqrt{567}$, $\sqrt{2{\cdot}75}$, $\sqrt{0{\cdot}52}$, $\sqrt{6} - \sqrt{3}$, $\sqrt{12} + \sqrt{8}$, $6\sqrt{26} - 4\sqrt{22}$.

2. Rationalize the denominators of the following fractions and then find their values to three significant figures:—

$$\frac{1}{\sqrt{2}},\ \frac{1}{2\sqrt{3}},\ \frac{8}{\sqrt{14}},\ \frac{1}{\sqrt{2{\cdot}1}},\ \sqrt{\frac{3}{13}},\ \sqrt{\frac{8}{11}},\ \frac{0{\cdot}8}{\sqrt{3{\cdot}9}},\ \frac{2\sqrt{5{\cdot}4}}{\sqrt{0{\cdot}22}}$$

3. Throw the following expressions into the form most suitable for calculation but do not actually find their values:—

$$\frac{\sqrt{24}}{\sqrt{3}},\ \frac{\sqrt{19}}{\sqrt{95}},\ \frac{\sqrt{18}}{\sqrt{24}},\ \frac{3\sqrt{5}}{2\sqrt{3}},\ \frac{4\sqrt{13}}{26\sqrt{2}},\ \frac{a}{\sqrt{2}},\ \frac{5\sqrt{p}}{4\sqrt{5}},\ \frac{a\sqrt{13}}{\sqrt{39}}$$

4. Calculate the value of the following expressions when $a = \sqrt{5}$, $b = \sqrt{2}$, $c = \sqrt{3}$:—

(i) $2a^2 - b^2$, (ii) $\frac{1}{3}(a^2 - bc)$, (iii) $a^2b + b^2c + c^2a$,

(iv) $\dfrac{a^2 + b^2}{2c}$, (v) $\dfrac{a - b}{5c^2}$, (vi) a^3,

(vii) $\dfrac{a^2 - b^2}{c^3}$, (viii) $\dfrac{a^5}{b^2c}$, (ix) $\dfrac{bc}{a^3}$,

(x) $(a + b)\,c$.

Note.—The answers to the following problems are to be left in a surd form.

5. The sides of a square are of length a. What is the length of the diagonal?

6. Bisect an equilateral triangle by a perpendicular drawn from one of the vertices to the opposite side. Call the length of half the base a. What is the length of the perpendicular?

7. One of the adjacent sides of a rectangle is three times the other. Call the length of the smaller a. Give an expression for the length of the diagonal.

8. A ladder of length $7a$ is placed against a house with its foot $3a$ away from the wall. How high is the top of the ladder above the ground?

9. (i) A spider comes out of a hole in the corner of a room, runs 6 feet along the foot of the wall, then climbs vertically up the wall which is 10 feet high, and having reached the ceiling crawls 7 feet out along it at right angles to the wall. How far is it from the hole?

(ii) Give a formula for the distance in all similar cases, calling the three movements a, b, and c.

(iii) If he spins a thread and descends from the ceiling a distance h, what is now his distance from the hole?

10. An aeroplane is above my head at a height of a feet. It flies due north and upwards until it is over a point p feet away and at a height of b feet. It then flies eastwards at the same level a distance of q feet. How far is it now from me?

11. A carriage wheel of radius r picks up a piece of paper from the road and carries it round one-twelfth of a revolution. How high is it now above the ground?

How high will it be when it has gone $\frac{7}{12}$ of the way round?

EXERCISE IX.

APPROXIMATION-FORMULÆ (I).

A.

1. Draw a square and call the length of its side a. Convert it into a square of side $a + b$ by the addition of two identical rectangles and a square. Write inside each figure the expression for its area.

2. Use the figure of No. 1 to complete the identity

$$(a + b)^2 =$$

3. (i) Calculate the area of each of the rectangles and of the square required to convert a square measuring 10 inches each way into one measuring 12 inches each way. Show that the resulting square has the correct area.

   (ii) Repeat the calculations, substituting the measurements 3·6 cms. and 4 cms.

   (iii) Repeat, substituting the measurements 14 cms. and 14·7 cms.

4. In 3 (i) what fraction of the side of the completed square (12 inches) is the added portion (2 inches)? What fraction of the area of the complete square (144 square inches) is the area of the added square (4 square inches)?

Answer the same questions with regard to 3 (ii) and 3 (iii). What do you notice about the two sets of fractions? Will this property always hold good?

5. I want to cover with linoleum a room 12 feet square. I have a square of linoleum measuring 11 feet each way and also a strip 22 feet long and 1 foot wide. What fraction of the whole floor space must I leave uncovered?

6. A square shed occupies the corner of a square playground. The length of its wall is one-twentieth of the length of the playground. What fraction of the area of the playground does it cover?

7. A square of length a is converted into one of length $a + b$ by adding a "gnomon" of width b. Compared with a, b is so small that the square b^2 may be neglected. What

formula for the increased area of the square may in this case be used instead of $A = (a + b)^2$? Throw it into the form most convenient for calculation.

Note.—A calculation in which you take account of a relatively small number b but neglect its square is said to be carried to a **first approximation.** The word "approximately" when used in this and the next exercise means "to a first approximation".

In writing a statement of an approximation, whether in symbols or in numbers, you should use, instead of the sign "=," the sign "$\doteqdot$". This symbol means "is approximately equal to".

8. (i) A square lawn 40 feet long has a path 3 feet wide along two sides. Use the formula of No. 7 to calculate approximately the area of the garden.
   Repeat the calculations, replacing the former dimensions
   (ii) by 53 feet and $3\frac{1}{4}$ feet;
   (iii) by 43 feet 6 inches and 3 feet 3 inches respectively.
9. (i) A square lawn 50 feet long has a path $2\frac{1}{2}$ feet wide all round it. Calculate the area of the garden to a first approximation. *About* what fraction of the whole area is neglected?
   (ii) Repeat the calculation, replacing the former dimensions by 77 feet and $3\frac{1}{2}$ feet respectively.

10. A metal tray is to be made a inches square with vertical sides b inches deep abutting directly on to one another without overlapping. The metal plate weighs w ounces per square inch. Write a formula (convenient for calculation) giving the total weight (W) of the tray.

Write also a formula for the percentage waste of metal when the tray is cut out of a square sheet measuring $a + 2b$ inches each way.

11. A square mirror measuring 17 inches each way is mounted in a frame $\frac{3}{4}$ inch wide. Find approximately the area of wall covered by the mirror and frame.

12. (i) Write down a formula for the exact increase in area in the square of No. 7.
    (ii) Write a formula which will give the increase to a first approximation.
13. (i) A square sheet of paper measuring 24 cms. in the side stretches 3 mms. each way when wetted. Find approximately the increase of its area.
    (ii) About what fraction of the whole area is being neglected?

(iii) What fraction of the *increase* of area is being neglected ?

(iv) Which of these fractions would you give if you were asked how exact your answer is ?

14. Draw (or, if it is more convenient, draw to half-scale) a rectangle 3 cms. high and equal to the total increase of area of the wetted paper in No. 13. Mark off a square equal to the part neglected in taking the first approximation. (The diagram will help you to realize its relative unimportance.)

15. The number 10^2 can be regarded as $(9 + 1)^2$, $(7 + 3)^2$, $(5 + 5)^2$, $(4 + 6)^2$, etc. Show that when the identity of No. 2 is applied to these expressions the result is 100 in each case. Choose another example of your own and find whether the identity works when applied to it.

16. Take any one of the cases in No. 15 and show by analysis *why* the identity holds good. Describe the steps of your analysis in symbols. What use do you now consider that you are entitled to make of the identity?

17. A piece of thin brass tube has an internal radius r and a thickness t. Write down a formula for calculating to a first approximation the area (A) of the ring of metal.

18. A metal ball has a thickness t and an internal radius r. Calculate approximately the area (A) of its external surface. How much greater is it than the internal surface ? [The area of a sphere of radius r is given by the formula $A = 4\pi r^2$.]

19. A boiler consists of a cylinder of length l and radius r, capped at each end by a hemisphere. The thickness of the metal is t. Show that the area of the outside surface is given approximately by the formula—

$$A = 2\pi\{(2r + l)(r + t) + 2rt\}$$

20. Complete the following identities :—

(i) $(a + 2b)^2 =$;	(ii) $(3p + q)^2 =$;
(iii) $(p + \frac{1}{2}q)^2 =$;	(iv) $(a + \sqrt{5}b)^2 =$;
(v) $\left(1 + \frac{p}{2}\right)^2 =$;	(vi) $(1 + 0{\cdot}003t)^2 =$;
(vii) $(1 + ct)^2 =$;	(viii) $(a + pb)^2 =$;
(ix) $\{l(a + b)\}^2 =$;	(x) $\{r(1 + \frac{1}{2}pt)\}^2 =$.

21. Suppose a straight line 1 unit long to be drawn anywhere on a sheet of metal, and let the metal then be heated till it is one degree warmer. The line will increase in length by a definite but very small amount c (called the "coefficient

of linear expansion"). If it is warmed 2°, unit length increases $2c$, and so on. Write formulæ for calculating :—

(i) The length (l) of an original unit when the metal is warmed t degrees.
(ii) The length (l_t) of a line originally 4 units long when the metal is warmed t degrees.
(iii) The same, the original length having been l_o.

22. A unit square is marked out on a sheet of metal which is then warmed t degrees. Find, to a first approximation :—

(i) The area (A_t) of the original unit square ;
(ii) The area (A_t) of a square which originally measured l units each way ;
(iii) The area of a piece of metal originally containing A_o square units.

23. Find to a first approximation the increase of area (I_t) of a circular disc of metal of radius r when it is warmed $t°$, the coefficient of linear expansion being c.

Use your formula to find the increase in area of a metal disc of radius 21·7 cms., when warmed 50°, the coefficient of expansion being 0·00006.

B.

24. Suppose you want to know *approximately only* the length of the side of a square containing 20 square cms. It is obviously between 4 cms. and 5 cms. If you take a square of 16 square cms. away from the original the residue will have an area of $20 - 16 = 4$ sq. cms. Arrange it as a long strip. In calculating its height to a first approximation what will you ignore? What then is the approximate height of the strip? The approximate length of the side of the square? Find out how far the square of your result differs from 20.

25. Find approximately (i.e. to one decimal place or to two if the second is a 5) the square roots of the following numbers : 10, 18, 54, 110, 175, 27, 410, 17·6, 83·5.

26. Calculate approximately the radii of the circles whose areas are (i) 13 square cms. ; (ii) 40 square inches ; (iii) 125 square feet.

27. Write down identities by means of which the values of the following expressions may be calculated approximately. (The second quantity mentioned is in each expression considerably smaller than the first) :—

(i) $\sqrt{a^2 + p} =$; (ii) $\sqrt{(a^2 + 2p)} =$;
(iii) $\sqrt{(a^2 + b^2)} =$; (iv) $\sqrt{r^2 + 1} =$;
(v) $2\sqrt{p^2 + q^4} =$; (vi) $\sqrt{16 + a} =$;
(vii) $\sqrt{81 + nt} =$.

28. Deal similarly with the following expressions :—

(i) $\sqrt{13 + 2a} =$; (ii) $\sqrt{24 + p} =$;
(iii) $\sqrt{a + b} =$; (iv) $\sqrt{p + 2q} =$;
(v) $\sqrt{(a + 2\sqrt{b})} =$; (vi) $\sqrt{(4p + 3\sqrt{q})} =$;
(vii) $\sqrt{(9a + 6\sqrt{b})} =$; (viii) $\frac{1}{8}\sqrt{(a + 3\sqrt{b})} =$.

29. (i) You start from a point H feet above the sea and climb a further height of h feet. Write a formula for D, the radius (in miles) of the circle of sea visible from the higher point. (See Ex. IV, No. 6.)

(ii) Suppose that h is small compared with H. Write a formula suitable for calculating D when $\sqrt{H}$ is known.

30. A boy standing on a cliff 400 feet above the sea climbs a tree 20 feet high so as to see as far as possible. (i) What is the greatest distance visible from the higher point? (ii) What additional distance was he able to see by climbing the tree? Write a formula for d the additional distance rendered visible by climbing h feet up from a point H feet above the sea—h being small compared with H.

EXERCISE X.

APPROXIMATION-FORMULÆ (II).

1. Draw a square measuring a each way. Mark off in one corner a smaller square measuring $a - b$ each way. Indicate in the gnomon two rectangles each of area ab. Suppose that you begin to reduce the larger square to the smaller square by taking away one of the rectangles ab; what must you do before you can take away the other?

Complete the identity $(a-b)^2 =$.

2. The length of the side of a square is reduced from a to $a-b$. Give a formula for calculating approximately the area of the reduced square when b is small compared with a.

3. The outside measurement of a square picture frame is 26·4 inches. The frame is 0·7 inches deep. Use the formula of No. 2 to find the approximate area of the picture.

4. A square sheet of paper measures 31·2 cms. each way when wet. In drying it shrinks 6 mms. each way. Find to a first approximation its area when dry.

Note.—Imagine the following. There are a number of piles of atlases on a shelf in this room, each pile containing nine. You are asked to carry 20 atlases into a neighbouring room and place them in piles on a table. By mistake you take away 4 complete piles, i.e. 9×4 atlases. Noticing your mistake you carry back 4 from each pile, i.e. 4×4, and put them on the shelf again. Thus you have removed altogether just as many as if you had originally taken 5×4, i.e. $(9 - 4) \times 4$ atlases. It is clear, then, that

$$\left.\begin{array}{c}\text{the total no. on shelf}\\ -\ (9-4)\times 4\end{array}\right\} = \left\{\begin{array}{l}\text{the same no.}\\ -\ 9\times 4 + 4\times 4.\end{array}\right.$$

5. Show that the identity of No. 1 holds good of all numbers by first considering $(9 - 4)(9 - 4)$ and then analysing the steps by symbols.

6. Complete the identities:—

(i) $(a - 2b)^2 =$; (ii) $\left(\frac{a}{2} - b\right)^2 =$;
(iii) $(p^2 - q^2)^2 =$; (iv) $(1 - p^2)^2 =$;
(v) $(1 - 2p^2)^2 =$; (vi) $(1 - ct)^2 =$;
(vii) $A(1 - ct)^2 =$; (viii) $\{r(1 - ct)\}^2 =$;
(ix) $\{a(1 - bt^2)\}^2 =$; (x) $(r^2 - \frac{1}{2}ab)^2 =$.

7. The outer radius of a thin metal organ pipe is r inches, the thickness t inches; write a formula for calculating approximately the volume (V) of the metal employed, considering the pipe as a cylinder of length l feet. Write also a formula for the cost (C) in shillings of the material employed, if 1 lb. contains v c. inches and costs c pence.

8. A sheet of metal of area A is cooled $t°$, the coefficient of expansion being c. Give an approximation-formula (i) for its area after cooling; (ii) for d, the decrease in its area. (Compare Ex. IX, No. 22.)

9. Write a formula for the area of the total surface of a closed metal cone, the radius being r and the slant height l. Find approximately how much it decreases when the cone is cooled $t°$, the coefficient of expansion being c.

10. Show by the figure of No. 1 that $\sqrt{a^2 - p} = a - \dfrac{p}{2a}$ approximately. (Compare Ex. IX, No. 24.)

11. Write down approximate equivalents to the following expressions:—

(i) $\sqrt{9a^2 - b}$; (ii) $4{\cdot}3\sqrt{\left(p^2 - \frac{q}{3}\right)}$;
(iii) $\sqrt{(16 - 2{\cdot}1t^2)}$; (iv) $\sqrt{a^2 - b^2}$;
(v) $\sqrt{(a - 4\sqrt{b})}$.

12. The bob of a pendulum of length a is pulled out a small distance b from the vertical. Write a formula for calculating approximately how much the bob rises.

Apply your formula to the case of a pendulum 2 feet long pulled 4 inches out of the vertical.

EXERCISE XI.

APPROXIMATION-FORMULÆ (III).

A.

1. What formula would you use to calculate $(a+b)^3$ to a first approximation, b being small compared with a?

2. A cubical biscuit tin measures 10 inches each way inside. The metal is $\frac{1}{12}$ inch thick. Calculate the volume of the metal exclusive of the solder and the side pieces of the lid.

3. An inch cube is heated $t°$. Find approximately its new volume, the coefficient of linear expansion being c.

4. Write a formula for calculating approximately the increase in volume, I, of a sphere of radius r when it is heated $t°$, the coefficient of linear expansion being c. $[V = \frac{4}{3}\pi r^3.]$

5. Show by the use of symbols that the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ can always be used.

6. Complete the following identities sufficiently for approximate calculation—the second number described being small compared with the first:—

(i) $(a + 2b)^3 =$; (ii) $(2a + b)^3 =$;

(iii) $\left(1 + \frac{p}{3}\right)^3 =$; (iv) $(a + pt)^3 =$.

7. (i) Complete the identity $(a^2 - 2ab + b^2)a =$.
(ii) Complete the identity $(a^2 - 2ab + b^2)b =$.
(iii) Complete the identity $(a^2 - 2ab + b^2)(a - b) =$.

(Read the note before Ex. X, No. 5.)

8. How could the cube model have been modified so as to show the existence of the identity

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3?$$

9. Complete the following identities sufficiently for approximate calculations—the second number being small compared with the first:—

(i) $\left(a - \frac{1}{2}b\right)^3 =$; (ii) $\left(\frac{p}{2} - q\right)^3 =$;

(iii) $(1 - 2{\cdot}3k)^3 =$; (iv) $\left(a - \frac{nt}{2}\right)^3 =$.

10. Show by the model how to find approximately the cube root of a number which does not differ much from a number the cube root of which is already known.

11. Complete for approximate computation the identities

(i) $\sqrt[3]{a^3 + d} =$; (ii) $\sqrt[3]{a^3 - d} =$.

12. Express each of the following numbers in one of the forms of No. 11 and find its cube root to a first approximation: 10, 30, 60, 1030, 322, 0·528, 0·023, 1·674.

B.

13. An inch-cube of metal is warmed $t°$; find its new volume, the (very small) coefficient of linear expansion being c.

Also find its volume after being cooled $t°$.

14. A sphere of the metal of No. 13 has a radius r. Find the increase in its volume when it is warmed $t°$.

15. A cube of metal measuring 1 inch each way is heated until its edges are 1·002 inch long. What is now (i) the area of each of its faces; (ii) its volume?

16. A cube of metal measuring 1 inch each way is cooled until its volume is reduced to 0·991 c. inch. What is now (i) the length of an edge; (ii) the area of a face?

17. The volume of a piece of metal at different temperatures is given by the formula $V = V_0 (1 + 0{\cdot}00018t)$, V_0 being its volume at 0°. Write down formulæ connecting (i) the length, L, of a rod of the metal with L_0 its length at 0°, and (ii) the area, A, of a plate of the metal with A_0 its area at 0°.

18. Given that $\sqrt[3]{3} = 1{\cdot}44$, $\sqrt[3]{5} = 1{\cdot}71$, find the value of $\sqrt[3]{24}$, $\sqrt[3]{40}$, $\sqrt[3]{81}$, $\sqrt[3]{320}$, $\sqrt[3]{15}$, $\sqrt[3]{25}$, $\sqrt[3]{120}$, $\sqrt[3]{45}$.

19. Express the following in the form most suitable for computation:—

$$\sqrt[3]{16},\ \sqrt[3]{108},\ \sqrt[3]{2160},\ \sqrt[3]{9000},\ \sqrt[3]{1{\cdot}6},\ \sqrt[3]{0{\cdot}648},\ \sqrt[3]{0{\cdot}0875}.$$

20. The following expressions describe certain combinations of numbers symbolized by the letters a, b, etc. Write down expressions which indicate the easiest way of finding approximately the cube root of each kind of combination: a^3b, ab^3, a^2b^3, a^4b, ab^4, a^2b^4, $a^2b^2c^3$, $a^3b^3c^2$, a^5b^3c, $a^2b^5c^4$, $27p^3r$, $108mn^3$, $8(p - q)$, $16(a^2 - b^2)$, $64p^3 (1 - 8t)$, $(a^4b + a^3)$, $(p^3 - p^3q^3)$, $(a^4b^3 - a^3b^4)$.

EXERCISE XII.

FRACTIONS (I).

1. Convert the following formulæ (in which the subject is $1/R$) into formulæ in which the subject is R :—

(i) $\frac{1}{R} = \frac{1}{p} + \frac{1}{q}$.

(ii) $\frac{1}{R} = \frac{1}{p} - \frac{1}{q}$.

(iii) $\frac{1}{R} = \frac{2}{p} + \frac{3}{q}$.

(iv) $\frac{1}{R} = \frac{3}{p} - \frac{5}{q}$.

(v) $\frac{1}{R} = 1 - \frac{p}{q}$.

(vi) $\frac{1}{R} = \frac{p}{q} - 2$.

(vii) $\frac{1}{R} = \frac{p}{q} + r$.

(viii) $\frac{1}{R} = r - \frac{q}{p}$.

(ix) $\frac{1}{R} = \frac{p}{q} + \frac{1}{r}$.

(x) $\frac{1}{R} = \frac{2p}{q} - \frac{3}{r}$.

(xi) $\frac{1}{R} = \frac{2q}{r} - 3p$.

(xii) $\frac{1}{R} = \frac{1}{r} - pq$.

Note.—The formulæ of Nos. 2 to 5 are to be thrown into the shape most convenient for substitution.

(If the numerator or denominator of a fraction is in the form $ab + ac$, or $a^2 - b^2$ will it be more convenient to leave it in that form or to factorize it ?)

2. (i) $A = \frac{2}{a} - \frac{6}{b}$.

(ii) $\frac{1}{A} = \frac{ab}{c} + a$

(iii) $P = p - \frac{pq}{r}$.

(iv) $\frac{1}{D} = d_1 + \frac{d_1 d_2}{d_3}$.

(v) $\frac{1}{B} = a + \frac{a}{b}$.

(vi) $\frac{1}{R} = \frac{r_1}{r_2} - 2r_1$.

(vii) $\frac{1}{R} = \frac{2r_1}{3r_2} - r_1$.

(viii) $A = ab + \frac{2b}{a}$.

(ix) $\frac{1}{A} = \frac{a}{b} - 3ab$.

(x) $A = \frac{a}{b} - \frac{b}{a}$.

(xi) $V = \frac{2u}{v} - \frac{v}{2u}$.

(xii) $\frac{1}{R} = \frac{4p}{q} - \frac{9q}{p}$.

3. (i) $P = \frac{1}{p} + \frac{1}{pq}$.

(ii) $P = \frac{2}{pq} - \frac{3}{p}$.

(iii) $P = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}$.

(iv) $\frac{1}{P} = \frac{1}{p} + \frac{1}{q} - \frac{2}{pq}$.

(v) $\frac{1}{P} = \frac{a}{p} + \frac{b}{q} - \frac{c}{pq}$.

4. (i) $V = \dfrac{1}{\frac{1}{u} + \frac{1}{v}}$.

(ii) $V = 2/\left(\frac{1}{u} - \frac{1}{v}\right)$.

(iii) $V = \dfrac{w}{\frac{1}{u} - \frac{1}{v}}$.

(iv) $V = \dfrac{a}{1 - \frac{v^2}{4u^2}}$.

(v) $V = a/\left(\frac{u^2}{v^2} - b^2\right)$.

5. (i) $V = uv\left(\frac{1}{u} + \frac{1}{v}\right)$.

(ii) $V = u\left(1 - \frac{v}{u}\right)$.

(iii) $V = v\left(\frac{1}{u} - \frac{1}{v}\right)$.

(iv) $V = \frac{u}{v}\left(1 - \frac{v^2}{u^2}\right)$.

(v) $V = \dfrac{v - u}{\frac{1}{u} - \frac{1}{v}}$.

(vi) $V = (u + 1)/\left(1 + \frac{1}{u}\right)$.

(vii) $V = (u - v)/\left(1 \quad \frac{v}{u}\right)$.

(viii) $V = \dfrac{\frac{1}{u} + \frac{1}{v} - \frac{1}{uv}}{u + v - 1}$.

(ix) $V = \frac{u - v}{u + v}\left(1 + \frac{v}{u}\right)$.

(x) $V = \left(1 - \frac{v^2}{u^2}\right)/(u + v)$.

(xi) $V = (u - v)/\left(\frac{u^2}{v^2} - 1\right)$.

(xii) $V = \frac{u}{v}/\left(\frac{u^2}{v^2} - a^2\right)$.

Find simpler expressions by which the following may be replaced when they occur in formulæ :—

6. (i) $\frac{1}{a^2} + \frac{1}{ab}$.

(ii) $\frac{1}{b^2} - \frac{1}{ab}$.

(iii) $\frac{1}{a^2b} - \frac{1}{ab^2}$.

(iv) $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$.

(v) $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}$.

(vi) $\frac{a}{bc} - \frac{c}{ab}$.

(vii) $\frac{1}{a^2} + \frac{2}{ab} + \frac{1}{b^2}$.

(viii) $\frac{1}{a^2} - \frac{2}{ab} + \frac{1}{b^2}$.

7. (i) $\frac{1}{3p^2} - \frac{1}{2p}$.

(ii) $\frac{a}{pb} + \frac{b}{pa}$.

(iii) $\frac{m}{an} - \frac{n}{am}$.

(iv) $\frac{1}{p^2a^2} + \frac{2}{pqab} + \frac{1}{q^2b^2}$.

(v) $\frac{2p}{3q^2} - \frac{3q}{2p^2}$.

8. (i) Two rectangular rooms are thrown into one room. The area of one of the rooms is A_1, of the other A_2. The breadth of the first room is b_1, of the second b_2. Find the length of the combined rooms;

(ii) The same, given that the area of the first is n times the area of the second room (A).

9. A man has a number of envelopes to address. The work would take him 5 hours. He engages the help of a boy who could address the whole of the envelopes in 7 hours. How would you find how long they would take together?

(i) Write a formula for calculating most easily the joint time (T), given the time taken by the man (m) and the boy (b) respectively when working alone.

(ii) Write a formula for the joint time if there are 2 men and 3 boys at work on the addressing.

(iii) Find the joint time taken by a man and a boy when the boy, working alone, would take n times as long as the man (T, n, m).

(iv) Find the joint time taken by a man and a boy given that the man, working alone, could do the work in $1/n$th of the time taken by the boy (T, n, b).

(v) Find formulæ for the joint time in each of the last two cases supposing that there are p men and q boys engaged on the work.

10. (i) When p equal marbles are dropped into a cylindrical jar of water the level of the water rises a height a. When q equal marbles of another size are dropped into an identical jar the change of level is b. How many times (n) is one of the former marbles as large as one of the latter?

(ii) The same, supposing that the area of the water surface in the second jar is 3 times as great as it is in the first;

(iii) The same, the radius of the first jar being 3 times as great as that of the second;

(iv) The same, s times the sectional area of the first jar being the same as t times the sectional area of the second ;

(v) The same, s times the diameter of the first being equal to t times the diameter of the second jar.

11. A bath has two taps. One is found to deliver a gallons of water in p minutes, the other b gallons in q minutes. Write formulæ for calculating :—

(i) The quantity (V) which is delivered into the bath in t minutes when both taps are running ;

(ii) The quantity delivered in a time which is n times q minutes ;

(iii) The time (T) which the first tap alone would take to run in Q gallons ;

(iv) The time which both taps together would take to run in Q gallons ;

(v) The time necessary to increase the quantity in the bath from Q_0 to Q gallons by the second tap only ;

(vi) The same when both taps are running.

12. A boat is rowed upon a lake for a distance of $1/p$ of a mile from east to west and then for $1/q$ of a mile to the north. How far is it from its starting-point?

13. A boy rows a boat for $1/p$ of a mile towards the east. Then turning round, he rows for $1/q$ of a mile in a direction between west and north until he is exactly due north of his starting-point. How far away is the starting-point?

14. Two cyclists ride together round a circular track, one along the outside edge where the radius is R, the other along the inside edge where the radius is r. A single revolution of the pedals carries the former's bicycle forward P feet, and the latter's p feet. Write down expressions for calculating :—

(i) The difference (d) between the number of pedal-revolutions made by the cyclists in going n times round the track ;

(ii) How many times (t) the pedal-revolutions of the former will be as numerous as the pedal-revolutions of the latter.

15. Two boys, X and Y, find that they can cover 100 yards in p paces and q paces respectively. X takes m paces in walking from the centre to the outer circumference of the track of No. 14 ; Y takes n paces in walking from the centre to the inner circumference. Write formulæ for :—

(i) The width (w) of the track in feet ;

(ii) The difference (d) in yards between the outer and inner circumferences ;

(iii) The difference (d) in miles between 200 revolutions along the outer and 200 revolutions along the inner circumference;

(iv) The area (A) of the track in square feet;

(v) The cost (C) in pounds of preparing the track at c pence per square yard.

16. (i) V c.cms. of water are poured from a narrow cylindrical jar into a wider one, the areas of cross-section being respectively A_1 and A_2. Find how many centimetres (d) the level of the water sinks;

(ii) The same, given that the radii of the jars are r_1 and r_2 respectively.

17. (i) In the preceding question if, in pouring the water from one jar to the other, $1/p$ of it was spilt, what fraction would remain?

(ii) To what height would it fill a jar of sectional area A?

(iii) To what height would it fill a jar of radius r?

(iv) What would be the answers to (ii) and (iii) if q/p of the water were spilt in transference?

18. V c.cms. of water are poured from a jar of sectional area A into a jug, and then poured back again into the jar. At each pouring q/pths of the water are spilt. Show how to calculate:—

(i) How much water (v) is returned to the jar;

(ii) How much lower (d) the level is than at first.

19. Instead of being poured back into the original jar the water is passed into a second cylinder of *smaller* sectional area, a, q/pths being lost at each passing as before. Find how much higher (d) the water is here than it was in the original cylinder.

20. (i) Taking the data of No. 19, find how many times (n) the water in the second cylinder is as high as it was in the first cylinder;

(ii) Answer the same question, given that the radii of the cylinders are R and r.

EXERCISE XIII.

FRACTIONS (II).

A.

1. Complete the following identities with a view to easier evaluation :—

(i) $16 - (a + 7) =$; (ii) $16 - (a - 7) =$;
(iii) $3a - (2a + 5) =$; (iv) $5p - (3p - 8) =$;
(v) $5p + (3p - 8) =$; (vi) $24 + (18 - r) =$;
(vii) $24 - (18 - r) =$; (viii) $24r + (18 - r) =$;
(ix) $24r - (18 - 6r) =$; (x) $(3p + 4b) - (2p - 4b) =$;
(xi) $a^2 - (p^2 - 3a^2) =$; (xii) $(10m^2 + p^2) - (m^2 + 5p^2) =$;
(xiii) $13 - (18 - 3r) =$; (xiv) $3p - (5p - 20) =$.

2. Write in full the explanations of your answers to No. 1 (ii), (vi), and (xiii) which you would give to a person who did not know how the results were obtained.

3. What restrictions are there upon the values of numbers that can be represented by the symbols in No. 1 (i), (ii), (vi), (xi), (xiii)?

4. Simplify the following expressions :—

(i) $14 - 2(7 - 3p)$. (ii) $19 - 3(5 - t)$.
(iii) $14t - 4(3t + 2)$. (iv) $3(a - 2b) + 2(a + 3b)$.
(v) $3(a - 2b) - 2(a + 3b)$. (vi) $5(2p - 3q) - 3(4p - 7q)$.
(vii) $6(3m - 2n) - 7(4m - 5n)$. (viii) $a(3p + 4) - 2ap$.
(ix) $a(3p + 4) - 5ap$. (x) $a(4 + b) - b(a - b)$.
(xi) $3(p + \sqrt{2}) - (2p + 5\sqrt{2})$.
(xii) $2(7\sqrt{2} - 4\sqrt{3}) - 4(3\sqrt{2} - 2\sqrt{3})$.

5. What restrictions are there upon the values of the symbols in No. 4 (v), (vi), (vii)?

6. Write full explanations of the answers which you give to No. 4 (vii) and (x).

7. (i) Owing to a decrease in the pressure a tap which usually delivers q gallons of water per minute into a tank actually supplies 8 gallons a minute less. Give the best formula for calculating how much longer (t) it will take to run in 100 gallons.

(ii) Adapt the formula to suit any given decrease of supply per minute (d) and any quantity of water to be supplied (Q).

8. When the tire is fully inflated the rear wheel of a bicycle has a radius of 14 inches. Find a formula for the extra number of revolutions per mile (n) when, owing to a leakage of air, the radius is reduced by a given amount (d).

9. In consequence of an increase in the duty on tobacco the retail price of cigarettes is to be raised by a penny the ounce packet, regardless of the quality of the cigarettes.

A certain smoker allows himself £3 per annum for cigarettes. How many ounce packets fewer must he buy in a year if he still purchases the same brand? Let $p \equiv$ "the old price of a packet in pence".

10. The smoker of No. 9 determines to increase by 7s. 6d. his annual allowance for cigarettes, but the larger sum does not yet permit him to buy as much tobacco as he formerly purchased for £3. Find how many packets short he is in the course of a year.

11. Throw the following formulæ into the shape most suitable for computation :—

(i) $\dfrac{5}{a} - \dfrac{5}{a + 6{\cdot}3}$. (ii) $1 - \dfrac{7{\cdot}2}{8{\cdot}4 - p}$.

(iii) $\dfrac{9{\cdot}8}{p + 6{\cdot}1} - 1$. (iv) $\dfrac{9{\cdot}8}{p - 6{\cdot}1} - 2$.

(v) $5{\cdot}2 - \dfrac{2{\cdot}2a}{3 + a}$. (vi) $\dfrac{5{\cdot}2}{a} - \dfrac{2{\cdot}2}{3 + a}$.

(vii) $a - \dfrac{2a}{0{\cdot}7a + 3}$. (viii) $\dfrac{3b}{7 - 3{\cdot}1b} - b$.

(ix) $\dfrac{1}{2{\cdot}3a} + \dfrac{1}{7{\cdot}7a - 5}$. (x) $8 - \dfrac{4p}{12 - \frac{3}{8}p}$.

(xi) $1 - \dfrac{1}{1 + \frac{1}{12}a}$. (xii) $\dfrac{\frac{3}{5}}{2 - \frac{1}{15}p} - 1$.

12. Simplify the following fractional expressions with a view to computation :—

(i) $\dfrac{1}{a} - \dfrac{1}{a + 2b}$. (ii) $\dfrac{1}{3p - q} - \dfrac{1}{3p}$.

(iii) $\dfrac{1}{5a} - \dfrac{1}{5a + b}$. (iv) $\dfrac{q}{3 - p} - \dfrac{3}{q}$.

(v) $\dfrac{pm}{m - n} - p$.

13. A dishonest tradesman measures off carpet for sale with a yard-measure which is really s inches short. Of how many pounds has he defrauded his customers by the time he has sold at a price of p shillings a yard a roll of carpet which really contains l yards?

14. A metal yard-measure is of the correct length at a temperature of 60° F. Obtain expressions for the error involved in using it to measure a length which is really l yards when the temperature is (i) $t°$ above 60°; (ii) $t°$ below 60°. (The coefficient of linear expansion of the metal is c.)

B.

15. Show that $\dfrac{1}{1-a} = 1 + \dfrac{a}{1-a}$

and that $\dfrac{a}{1-a} = a + \dfrac{a^2}{1-a}$

Hence show that $\dfrac{1}{1-a} = 1 + a + \dfrac{a^2}{1-a}$

16. Obtain a similar expression for $\dfrac{1}{1+a}$.

17. If a^2 is so much smaller than a that it may be neglected, what expressions may be substituted for $\dfrac{1}{1-a}$ and $\dfrac{1}{1+a}$ respectively?

18. Complete the following identities for the purpose of approximate calculation (the second member of each denominator is very small compared with the first):—

(i) $\dfrac{1}{1-pt} =$; (ii) $\dfrac{d}{1+pt} =$;

(iii) $\dfrac{d}{1+0{\cdot}0006t} =$; (iv) $\dfrac{1}{1-\frac{b}{a}} =$;

(v) $\dfrac{a}{1-\frac{b}{a}} =$; (vi) $\dfrac{\frac{1}{a}}{1+\frac{b}{a}} =$;

(vii) $\dfrac{1}{a\left(1+\frac{b}{a}\right)} =$; (viii) $\dfrac{1}{a+b} =$;

(ix) $\dfrac{r}{p-q} =$; (x) $\dfrac{a}{p^2+q^2} =$.

19. Write formulæ for the approximate solution of (i) No. 8, (ii) No. 14.

20. On the morning of a certain day Consols can be bought for £80 per £100 stock; at the end of the day the price has risen to £80½. Find approximately the difference in the amount of stock purchasable for £3200.

21. On account of a collision between two of their trains the stock of a railway company falls from 102 to 101¾ per £100. What difference, approximately, does this fall make to the amount of stock that can be bought for £5100?

22. At a temperature of 60° the length of a rail is l feet, but in laying the railway line space must be left between the rails to allow for the expansion due to a possible rise of temperature of (say) 40°. Find what difference this allowance makes to the number of rails used in laying down a line L miles long, the coefficient of linear expansion of the rails being c.

23. A steamer is timed to cover a certain journey of m miles at an average speed of s mls./hr. This speed is subject, according to the direction of the wind, to a relatively small change of d mls./hr. Find a formula for the difference in hours between the longest and the shortest journey.

24. In each of the following expressions the second member of a denominator is very small in comparison with the first. Write down a series of expressions which are equivalent to the given ones for the purpose of approximate calculation:—

(i) $\dfrac{1}{1 - at} - \dfrac{1}{1 - bt}$. (ii) $\dfrac{1}{a + bt} - \dfrac{1}{c + dt}$.

(iii) $\dfrac{1}{(1 + a)^2}$. (iv) $\dfrac{1}{(1 - a)^2}$.

(v) $\dfrac{1}{a^2(1 + pt)^2}$. (vi) $\dfrac{a^2}{(a + b)^2}$.

(vii) $\dfrac{1}{(p - q)^2}$. (viii) $\dfrac{1}{(p - q)^2} - \dfrac{1}{p^2}$.

(ix) $\dfrac{1}{\sqrt{(1 + a)}}$. (x) $\dfrac{1}{\sqrt{(a^2 - p)}}$.

(xi) $\dfrac{p}{\sqrt{(p^2 - r)}} - 1$. (xii) $\dfrac{1}{\sqrt{(100 - ct)}} - \dfrac{1}{10}$.

(xiii) $\dfrac{1}{\sqrt{(l^2 - ct)}} - \dfrac{1}{l}$. (xiv) $\dfrac{\sqrt{a}}{\sqrt{(a - b)}}$.

(xv) $\dfrac{1}{\sqrt{p}} - \dfrac{1}{\sqrt{(p + q)}}$. (xvi) $\dfrac{1}{(1 - a)^3}$.

(xvii) $\dfrac{1}{r^3(1 + ct)}$. (xviii) $\dfrac{a^3}{(a - b)^3}$.

(xix) $\dfrac{1}{\sqrt[3]{(1 - p)}}$. (xx) $\dfrac{1}{a} - \dfrac{1}{\sqrt{(a^3 + r)}}$.

25. A number of brass cylinders are to be made of exactly the same volume, V, and of approximately the same radius, r. It is found, when they are finished, that the greatest and least radii differ from r only by a small quantity a. Find h, the difference in height between the longest and the shortest of the cylinders.

26. The same, when the volume of the cylinder may in extreme cases be either greater or less than V by the small amount v.

27. The diameter of the cylinder of a marine engine is D inches. The formula of Ex. IV, No. 31, shows how to calculate the loss of steam pressure (P) due to friction of the piston against the cylinder. Obtain a formula for the difference in the loss of pressure (p) when the diameter of the cylinder is increased by an amount h inches, small compared with the whole diameter.

Find p when D = 81 inches and h = 1 inch.

28. Obtain a similar formula for p when the diameter of the cylinder is decreased by h.

29. Write a formula for N, the number of times a pendulum of length l swings in T seconds. (See Ex. VII, No. 11.)

The pendulum is made shorter by a small amount h. Write a formula by which to calculate approximately the additional number (n) of swings in T seconds.

30. Obtain a formula for the case in which the pendulum's length is increased instead of being decreased.

31. Obtain a formula for calculating $\dfrac{1}{1 - a}$ to a second approximation.

32. Obtain a similar formula for $\dfrac{1}{1 + a}$.

33. Obtain similar formula for

(i) $\dfrac{1}{a - b}$, (ii) $\dfrac{1}{a + b}$, (iii) $\dfrac{1}{1 + ct}$.

(The second member of each denominator is small compared with the first.)

C.

34. Express the following in forms more convenient for substitution :—

(i) $P = \dfrac{1}{p+2} + \dfrac{1}{p+3}$.

(ii) $P = \dfrac{1}{p+2} - \dfrac{1}{p+3}$.

(iii) $P = \dfrac{1}{p+a} + \dfrac{1}{p+b}$.

(iv) $\dfrac{1}{P} = \dfrac{1}{p+b} - \dfrac{1}{p+a}$.

(v) $\dfrac{1}{P} = \dfrac{1}{p-a} + \dfrac{1}{p+a}$.

(vi) $\dfrac{1}{P} = \dfrac{1}{2p-a} - \dfrac{1}{2p+a}$.

(vii) $Q = \dfrac{1}{n-a} + \dfrac{1}{m-a}$.

(viii) $Q = \dfrac{1}{n-a} - \dfrac{1}{m-a}$.

(ix) $\dfrac{1}{Q} = \dfrac{1}{qn+b} - \dfrac{1}{pm+a}$.

(x) $\dfrac{1}{Q} = \dfrac{m}{q-m} - \dfrac{n}{p-n}$.

(xi) $\dfrac{1}{Q} = \dfrac{p}{q-a} - \dfrac{q}{p-a}$.

(xii) $M = \dfrac{3a}{2m-3n} + \dfrac{2a}{3m-2n}$.

(xiii) $M = \dfrac{3a}{2m-3n} - \dfrac{2a}{3m-2n}$.

(xiv) $\dfrac{1}{M} = \dfrac{pa}{1-pa} - \dfrac{qb}{1-qb}$.

35. Reduce each of the following to a single equivalent fractional expression :—

(i) $\dfrac{1}{a+b} + \dfrac{1}{(a+b)^2}$.

(ii) $\dfrac{1}{a+b} - \dfrac{a}{(a+b)^2}$.

(iii) $\dfrac{a}{(a-b)^2} - \dfrac{1}{a-b}$.

(iv) $\dfrac{1}{a-b} + \dfrac{b}{(a-b)^2}$.

(v) $\dfrac{1}{a-b} + \dfrac{2b}{(a-b)^2}$.

(vi) $\dfrac{1}{a+b} + \dfrac{1}{a^2-b^2}$.

(vii) $\frac{1}{a-b} + \frac{b}{a^2-b^2}$.

(viii) $\frac{1}{a-b} - \frac{b}{a^2-b^2}$.

(ix) $\frac{1}{a+b} + \frac{1}{a-b} + \frac{1}{a^2-b^2}$.

(x) $\frac{1}{a-b} - \frac{1}{a+b} + \frac{1}{a^2-b^2}$.

(xi) $\frac{1}{a+b} + \frac{1}{a-b} + \frac{2b}{a^2-b^2}$.

(xii) $\frac{1}{a-b} - \frac{1}{a+b} + \frac{2a}{a^2-b^2}$.

(xiii) $\frac{1}{a-b} - \frac{1}{a+b} - \frac{2b}{a^2-b^2}$.

(xiv) $\frac{1}{a-b} + \frac{1}{a+b} - \frac{2a}{a^2-b^2}$.

(xv) $\frac{1}{(p+q)^2} + \frac{1}{p^2-q^2}$.

(xvi) $\frac{1}{p^2-q^2} - \frac{1}{(p+q)^2}$.

(xvii) $\frac{1}{(p-q)^2} - \frac{1}{p^2-q^2}$.

(xviii) $\frac{p}{(p-q)^2} - \frac{q}{p^2-q^2}$.

(xix) $\frac{q}{(p-q)^2} + \frac{p}{p^2-q^2}$.

(xx) $\frac{p}{(p+q)^2} + \frac{q}{p^2-q^2}$.

36. Simplify the following algebraic fractions :—

(i) $\frac{a}{a^2-b^2} - \frac{b}{(a+b)^2} + \frac{1}{a+b}$.

(ii) $\frac{a}{a^2-b^2} + \frac{b}{(a-b)^2} + \frac{1}{a-b}$.

(iii) $\frac{a}{a^2-b^2} + \frac{b}{(a+b)^2} - \frac{1}{a+b}$.

(iv) $\frac{a}{a^2-b^2} - \frac{b}{(a-b)^2} - \frac{1}{a-b}$.

(v) $\frac{1}{(a+b)^2} - \frac{a}{(a+b)^3}$.

(vi) $\frac{1}{a+b} - \frac{2ab}{(a+b)^3}$.

(vii) $\frac{1}{a-b} + \frac{a+b}{(a-b)^2} + \frac{2ab}{(a-b)^3}$.

(viii) $\frac{1}{a+b} - \frac{a-b}{(a+b)^2} - \frac{2ab}{(a+b)^3}$.

EXERCISE XIV.

CHANGING THE SUBJECT OF A FORMULA (I).

A.

Note.—Whenever your answer to a problem is a number you should test its correctness.

1. I am thinking of a number. I multiply it by 3·6 and add 14·7. The result is 23·18. What is the number?

2. I am thinking of a number. I multiply it by $3\frac{3}{8}$ and subtract $1\frac{1}{6}$ from the product. The remainder is $\frac{1}{3}$. Find the number.

3. I am thinking of a number. I add to it 2·3 and multiply the total by 8·9. The result is 133·5. What is the number?

4. I am thinking of a number. I subtract from it $\frac{3}{7}$ and multiply the remainder by $\frac{2}{5}$. The result is $\frac{1}{7}$. What is the number?

5. I am thinking of a number. I subtract 4·32 from it and divide the residue by 3·24. The quotient is 4. Find the number.

6. I am thinking of a number. I divide it by 4·4 and add 7·35 to the quotient. The result is 13·6. Calculate the number.

7. Write down in symbols a statement of the various steps followed in working out each of Nos. 1 to 6. Put $n \equiv$ "the number thought of" and use a, b, c as symbols for the other numbers, whether integers or fractions.

Note.—In the following six examples (Nos. 8 to 13) you are to calculate the number described either (1) mentally or (2) by setting down the statement in symbols and applying to it the rules by which a number may be moved from one side of the sign = to the other.

If you find the number mentally you must afterwards set down the steps by which you found it, and see whether they followed the rule.

8. I add 6 to twice a certain number and multiply the sum by 7. The product is 84.

9. Taking a certain number I diminish it by 5 and multiply the residue by 9, obtaining 99 as the product.

10. I multiply a certain number by 5, subtract 4 and divide the residue by 7. The quotient is 8.

11. A certain number is multiplied by 7, the product added to 12, and the sum divided by 4. The result is 17.

12. I think of a number. I add seven to it and divide it into 60. It goes 5 times.

13. I think of a number, and divide 72 by another number which is 5 less than double the former number. The quotient is 8.

14. Describe in symbols the steps followed in solving any three of the last six examples. Put n for the number to be calculated and a, b, c, d for the other numbers mentioned.

Note.—The following eight examples (Nos. 15 to 22) are statements of "Think of a Number" problems. Express the problems in words and solve them.

15. $$3{\cdot}4(7{\cdot}1n - 12{\cdot}8) = 53{\cdot}04.$$

16. $$\frac{6{\cdot}3n + 26{\cdot}15}{8{\cdot}3} = 5{\cdot}2.$$

17. $$\frac{14{\cdot}91}{(3{\cdot}7n - 4{\cdot}9)} = 7.$$

18. $$6(7n - 5) - 50 = 46.$$

19. $$\frac{3n + 8}{7} - 3 = 2.$$

20. $$\tfrac{1}{3}(4n - 9) + 18 = 25.$$

21. $$\frac{33}{5n + 7} + 18\tfrac{1}{2} = 20.$$

22. $$\frac{14{\cdot}8}{2n - 5} - 8{\cdot}3 = 6{\cdot}5.$$

23. Describe in symbols the steps by which you could solve any problems like (i) No. 17, (ii) No. 19, (iii) No. 21.

Write down in symbols the steps by which you would in each case calculate the number (n) described in the following eight examples (24 to 31).

24. $$\frac{an - b}{c} + e = d.$$

25. $$\frac{pn + q}{r} - a = b.$$

26. $\frac{1}{a}(bn - c) - p = q.$

27. $\frac{a}{pn + q} - b = bc.$

28. $\frac{a}{pn - q} + c = b.$

29. $p\left(\frac{n}{a} - q\right) - p = b.$

30. $p\left(\frac{n}{a} + q\right) = b.$

31. $\frac{a}{\frac{n}{p} + q} - b = c.$

B.

Note.—The formulæ of the following four examples are all taken from Ex. III. After changing the subject of one of these formulæ you should in each case express in words the meaning of the new formula derived from it, and make up your mind whether it is true.

32. Write down the formula of Ex. III 1 (ii). Obtain from it formulæ (i) for the number of marbles in the bag; (ii) for the weight of a single marble.

33. Write down the formula of No. 1 (vi). Obtain from it formulæ (i) with t_1 as the subject; (ii) with t_2 as the subject.

34. Write down the formula of No. 2 (ii). Change its subject to t.

35. Write down the formula of No. 4 (ii). Obtain from it formulæ (i) with S_0, (ii) with n, (iii) with t as subject.

Note.—The formulæ of Nos. 36 to 41 are taken from Ex. IV. Be prepared to explain the meaning of the formulæ you derive from them.

36. From the formula of No. 17 obtain formulæ (i) for l, (ii) for b.

Find the greatest length of wrought iron bar 4 inches broad and 3 inches deep that can safely support a weight of half a ton hung at the end of it.

37. The indicated horse-power of a marine engine is 150. The diameter of the cylinder is 5 feet, the length of piston stroke 4 feet 2 inches, the number of revolutions 30 per min.

Turn the formula of No. 33 into one from which the mean steam pressure can be calculated.

38. Derive from the formula of No. 4 a formula which will enable you to calculate the diameter of the chain-iron appropriate to a given load. Use it to calculate the diameter necessary for lifting 10 tons.

39. Change the subject of the formula of No. 5 from p to L. What does the new formula tell you?

40. Change the subjects of No. 25 (i) and (ii) to l. Do the results agree? What length of span is possible in the case of a wire weighing 0·12 lb./ ft., if it may not be stretched with a force greater than 256 lbs. wt. nor allowed to dip more than 8 inches.

41. Change the subject (i) of No. 6 to h; (ii) of No. 22 to d; (iii) of No. 31 to D.

Note.—The formulæ of Nos. 42 to 47 are taken from Ex. III.

42. Write the formula of No. 3 (ii). Change its subject to i.

43. From the formula of No. 15 (ii) obtain formulæ with (i) w_1, (ii) n as subject.

44. Change the subject of the formula of No. 28 (iv) from d to (i) d_0, (ii) t, (iii) s_1.

45. Find a formula for the total rainfall (R) during the years between two given dates (d_1 and d_2) given the monthly average (r).

46. Derive a formula for R from No. 30. Does it agree with the one you have just found?

47. Take the formula of No. 32 (i) and change its subject to (i) T, (ii) t, (iii) d_2.

Note.—The formulæ of Nos. 48 to 50 are taken from Ex. IV.

48. Change the formula of No. 24 into one by which you may calculate the height from which the rain should fall, given its weight, etc.

49. From the formula of No. 23 derive a rule for finding the weight that can safely be lifted by a crane hook, given the thickness of the shank.

50. In No. 25 show (i) that formula (iv) could be derived from (iii); (ii) that formula (iii) could be derived from (iv).

EXERCISE XV.

CHANGING THE SUBJECT OF A FORMULA (II).

A.

1. A certain number is multiplied by 2·3 and the product is taken from 12·1. The residue is 4·28. Find the number.

2. A humorist when asked his age replied, "Multiply my age by 6, subtract the product from 840, and divide the residue into 1899. Add 7 to the quotient and the answer will be 10." What was his age?

3. Asked the same question next day by another person, he replied, "Take 10 times my age away from 840 and divide the difference into 1530. Subtract the quotient from 10 and you will have 7 left." Does this answer agree with the former?

4. Divide a certain number by 26, subtract the quotient from 8·2, multiply the difference by 12 and you will obtain 20·4. What is the number?

5. What number would you obtain as the answer to No. 4 if, by mistake, you thought that you were to subtract 8·2 from the quotient by 26?

6. I take 4 times a certain number from 28·6 and subtract the result from 11·3. The residue is 3·5. Find the number.

7. If in doing No. 6 you had read "subtract three times the result" for "subtract the result," what answer would you have obtained?

8. Describe in symbols the steps taken in solving any two you please of Nos. 1 to 7.

Show how you would in each case find the number n described in Nos. 9 to 13.

9. $$a + \frac{b}{q - pn} = c.$$

10. $$a - \frac{1}{p(b - n)} = c.$$

11. $$a - p(b - cn) = q.$$

12. $$a^2 + b(p - q^2n) = c^2.$$

13. $$\frac{p}{q}(a^2 - b^2n) = r^2.$$

Note.—The formulæ in Nos. 14 to 17 are taken from Ex. III. See that you understand the meaning of the formulæ which you derive from them.

14. Write the formula of No. 27 (iii) and change the subject to b.

15. From the formula of No. 28 (v) obtain a formula for the speed of the slower car (s_2).

16. From the formula of No. 32 (iii) derive a formula for d_2, the second observed depth of the water.

17. Derive a formula for d_2 from that of No. 32 (iv).

The formulæ in Nos. 18 to 21 are taken from Ex. IV. Be ready to explain the meaning of the formulæ you derive from them.

18. In No. 3 obtain a formula for T.

19. Obtain from No. 15 a formula by which to solve the following problem: A person whose eye is 5 feet 6 inches above the ground sets up a pole 12 feet high and on standing back 14 feet can just see over the end of the pole the top of a church spire 180 feet high. He now moves the pole 80 feet nearer to the church. Where must he stand so as to see the spire as before?

20. Change the subject of the formula of No. 29 (i) to t; (ii) to n.

21. Change the subject of No. 25 (iv) to L. Does the new formula agree with 25 (iii)?

22. The following formula is taken from the Pocket Book of an electrical engineer. Derive from it a formula for l_1.

$$E = \tfrac{1}{2}(R_1 + R_2 + R_3) - l_1.$$

23. The same Pocket Book yields the following formula. Change the subject to d.

$$I = B \cdot \frac{D^3 - d^3}{12}.$$

B.

24. A certain number when multiplied by 6 gives the same result as when it is multiplied by 4 and 14·3 is added to it. What is the number?

25. A certain number when multiplied by 3·4 is equal to its product by 1·8 together with 6·4. Find the number.

26. What are the numbers described in the following statements?—

$$\begin{array}{rl}
\text{(i)} & 3n + 4 = n + 10. \\
\text{(ii)} & 4n - 7 = 11 - 2n. \\
\text{(iii)} & 28 - 3n = 43 - 6n. \\
\text{(iv)} & 7{\cdot}3 + 5n = 12n - 7{\cdot}4.
\end{array}$$

27. Describe in symbols the steps taken in solving problems like (ii) and (iv) of No. 26.

28. The humorist of Nos. 2 and 3, when shown that his answers did not agree, said, "I will tell you my real age. If you multiply it by 2 or if you divide it by 3 and subtract the quotient from 350 you get the same number". What was his age according to this statement?

29. My age is three times my son's. If you multiply his age by 5 and subtract 30 you will again have my age. How old are we?

30. A man who was asked by a foolish young sportsman to sell his horse said, "I will let you have it either for four times as much as I gave for it less £130 or for £220 less three times what I gave for it. Choose which price you will pay." As a matter of fact the seller knew that the two prices were the same. Find how much he gave for the horse and at what price he was offering it for sale.

31. I am thinking of a number. Three times the residue when it is subtracted from 81 is the same as four times the residue when 60 is subtracted from it. What is the number?

32. Find the numbers referred to in the following statements:—

$$\begin{array}{rl}
\text{(i)} & 8n = 3(n + 20). \\
\text{(ii)} & 7(n + 2) = 5(n + 8). \\
\text{(iii)} & 7(n + 4) = 11(n - 2). \\
\text{(iv)} & 4(n + 3{\cdot}4) = 3(14{\cdot}1 - n). \\
\text{(v)} & 8(20 - n) = 9(n - 3). \\
\text{(vi)} & 4(3n - 4) = 7(n + 2). \\
\text{(vii)} & 5(2n + 5) = 7(4n - 17). \\
\text{(viii)} & 5(4 - 3n) = 6(5 - 4n).
\end{array}$$

33. Describe in symbols the steps by which problems like (i), (iii), (v), and (vii) of No. 32 can be solved.

34. X had a friend C who went to live in a road in which X's friend A occupied No. 20 and his friend B No. 41. C gave X a rule for remembering his number. Unfortunately

X forgot whether the rule was "Three times the difference between my number and A's is the same as four times the difference between my number and B's," or whether the words "three" and "four" should be interchanged. What may C's number have been?

35. A man learnt that a casket of jewels had been hidden in a certain hedge and that twice its distance from one of two trees growing out of the hedge was three times its distance from the other. He found that the two were 60 yards apart. Calculate the various possible positions of the jewel case and show them in a diagram.

36. I am thinking of a number. One-fifth of the residue when it is subtracted from 100 is the same as one-sixth of the residue when 12 is subtracted from it. Find the number.

37. Find the numbers referred to in the following statements:—

(i) $\dfrac{n-2}{5} = \dfrac{50-n}{7}$.

(ii) $\dfrac{n+3}{3} = \dfrac{3n-1}{7}$.

(iii) $\dfrac{14-3n}{13} = \dfrac{4n-5}{10}$.

(iv) $\dfrac{51-2n}{3} = \dfrac{3n-5}{2}$.

38. Describe in symbols the steps to be taken in solving problems like (i) and (iv) of No. 37.

39. A bought a house for £360 and a year later sold it to B. When B left it he sold it to C for £480. B afterwards boasted that his profit was half as large again as A's. At what price did he sell the house?

40. What values of *n* will make the following statements true?—

(i) $\dfrac{2}{3}(n+7) = \dfrac{3}{5}(2n-3)$.

(ii) $\dfrac{3}{4}(28-5n) = \dfrac{2}{5}(4n-1)$.

(iii) $\dfrac{4}{7}(5n-3) = \dfrac{1}{3}(81-n)$.

(iv) $\dfrac{n}{2n-7} = \dfrac{5}{3}$.

(v) $\dfrac{4n-7}{7n+4} = \dfrac{5}{12}$.

(vi) $\dfrac{23 - 3n}{23 + 3n} = \dfrac{1}{22}$.

(vii) $\dfrac{3}{n + 3} = \dfrac{7}{3n - 1}$.

(viii) $\dfrac{4}{5n - 3} = \dfrac{9}{30 - n}$.

(ix) $\dfrac{3{\cdot}1}{3n - 9{\cdot}5} = \dfrac{2{\cdot}4}{2n - 4{\cdot}7}$.

(x) $\dfrac{51}{4n - 3} = \dfrac{39}{2n + 3}$.

41. A cyclist A rides a distance of 28 miles in the time that another cyclist B takes to ride 18 miles. My own usual rate of riding is 3 miles an hour less than A's and 2 miles an hour greater than B's. How fast do I ride?

42. Find the values of n which satisfy the following statements :—

(i) $8(71 - n) = 5(n - 14) + 1$.
(ii) $4(2 - n) + 3(2 + n) = 13$.
(iii) $7(2n - 3{\cdot}2) - 4(14{\cdot}7 - 4n) = 2{\cdot}5$.
(iv) $11n + 6{\cdot}5 = 42{\cdot}9 - 3(n - 12{\cdot}8)$.
(v) $4(28{\cdot}5 - 3n) - 3(2n - 11{\cdot}3) + 4{\cdot}5 = 2(48{\cdot}7 - 5n) - 3{\cdot}4$.
(vi) $5(2n - 7) = 28 - 2(2n - 7$.
(vii) $\dfrac{27 - 4n}{3} = \dfrac{27 - 4n}{4} + 2$.
(viii) $6(3n - 4) + 5(7 - 2n) - 12 = 6(7 - 2n) - 5(3n - 4) + 7$.

43. I am thinking of a number. The difference between one-third and one-sixth of the number is 5. What is the number?

44. Find a number the fourth part of which exceeds the fifth part by 3.

45. Find the values of n which satisfy the following statements :—

(i) $\dfrac{n}{3} - \dfrac{n}{4} = 4$.

(ii) $\dfrac{2}{3}n = 7 - \dfrac{3}{4}n$.

(iii) $\dfrac{3n}{5} + \dfrac{2n}{3} = 1$.

(iv) $2n = \dfrac{14n}{3} - 5$.

(v) $\dfrac{n + 19}{5} = \dfrac{n}{4} + 3$.

(vi) $\dfrac{2}{3}(11 - 2n) = \dfrac{1}{4}(3n - 2) - \dfrac{1}{2}$.

(vii) $\frac{n-7}{10} + \frac{n-3}{4} = \frac{n-2}{5}$.

(viii) $\frac{2n-3}{4} - \frac{7-n}{2} = \frac{3n-1}{5} - 1{\cdot}81$.

(ix) $\frac{7n}{5} - \frac{1}{10}(n-11) = \frac{3}{4}(n+4) + 1$.

(x) $5\left(\frac{3n-1}{3} - \frac{n+4}{4}\right) = \frac{1}{3} + n$.

46. The following are general descriptions of statements of the kinds given in the preceding examples. Exhibit the steps by which the value of n could be determined in the case of each form of statement :—

(i) $\frac{an}{bn-c} = \frac{p}{q}$.

(ii) $\frac{a}{pn-q} = \frac{b}{p-qn}$.

(iii) $a(pn+q) - b(rn-s) = c$.

(iv) $\frac{a}{n} = \frac{b}{n} + c(a+b)$.

(v) $a(pn-q) = b(pn-q) + c$.

(vi) $\frac{n-a}{p} + \frac{n-b}{q} = \frac{n-c}{r}$.

C.

47. From No. 46 (ii) derive formulæ for calculating p and q.

48. From No. 46 (iv) derive a formula for calculating a/b.

49. From No. 46 (v) derive a formula for calculating $a - b$.

Note.—The formulæ of Nos. 50-57 are taken from a book on electricity.

50. Change the subject of the following formula to h :—

$$c = c_2\frac{1+2h}{1-h}.$$

51. Change the subject of the following formulæ from i to V_2 :—

$$i = cS\frac{V_1 - V_2}{l}.$$

52. Obtain formulæ having h and a respectively as subject from the formula—

$$F = 4\pi I\left(1 - \frac{h}{a}\right).$$

53. Obtain a formula for n from the statement :—

$$c = \frac{L}{R}\cdot\frac{d}{b}(2b^2 - n^2).$$

54. Change the subject of the following formula to V_1 :—

$$Q = S_1 \frac{V - V_1}{4\pi}\left(\frac{1}{d_1} + \frac{1}{d_2}\right).$$

55. Find a formula for q, given that

$$a_0 = a\left(\frac{1}{1 - d} - q\right).$$

56. Convert the following formula into one with r_2 as subject :—

$$e = \mathrm{RI}\frac{r_1}{r_1 - r_2}.$$

57. Obtain an expression for V_1 from the formula

$$\frac{1}{R} = \frac{Cl}{t} \cdot \frac{V}{V - V_1}.$$

58. From the formula $e = A - \frac{B}{r^2}$ derive one with r as subject.

59. From the mechanical formula $Fs = W . \frac{v^2 - u^2}{2g}$ derive one with u as subject.

60. Change the subject of the following formula to r :—

$$ev = \frac{p_1 - p_2}{4}(a^2 - r^2).$$

EXERCISE XVI.

SUPPLEMENTARY EXAMPLES.

A. Formulation.

1. A certain sum is required to restore a church, and subscriptions are coming in a constant monthly rate. Write formulæ to find :—

(i) The amount still to be collected after a given number of months (S, S_0, m, t) ;

(ii) The number of months after this before the fund will be closed (T).

2. Given the area of a square plot of grass write a formula for the length of the side of the plot (s, A).

An oblong plot of grass in the middle of a garden is to be altered into a square by cutting off from its length and adding to its breadth, its area being unchanged. Write formulæ for :—

(i) The length of the side of the square, given the length and breadth of the oblong (s, l, b) ;

(ii) The amount to be cut off from the length (p) ;

(iii) The amount to be added to the breadth (q).

3. Write down formulæ for finding :—

(i) The circumference of a circle, given its diameter (C, d) ;

(ii) The circumference of a circle, given its radius (C, r) ;

(iii) The radius of a circle, given its circumference ;

(iv) The radius of a circle, given the number of times that it revolves in rolling through a certain distance (r, n, l) ;

(v) The number of times that a circle of given radius will revolve in rolling a given distance ;

(vi) The area of a circle, given its radius (A, r) ;

(vii) The radius of the circle, given its area ;

(viii) The volume of a cylinder, given its radius and height (V, r, h) ;

(ix) The height of a cylinder, given its volume and radius ;

(x) The radius of a cylinder, given its volume and height.

4. (i) Give a formula for finding how much the surface of water in a rectangular cistern will sink when a given number

of cubic feet of water are drawn off, the length and breadth of the cistern being given in feet (h, V, l, b);

(ii) The same, the cistern being cylindrical and of a given radius (r).

(iii) A gallon of water occupies 0·16 cubic feet. Find how much the water will sink in a rectangular cistern when the quantity drawn off is measured in gallons;

(iv) The same, the cistern being cylindrical.

5. (i) Two motor-cars start together on the same road but one travels faster than the other. How far will they be apart after a certain number of hours (d, s_1, s_2, t)?

(ii) Write a formula that shall show how far the slower has gone when the faster has gone a certain distance (d_2, d_1).

(iii) Write a formula to show how far the slower car is behind the faster when the latter has gone a certain distance (D_1, d_1).

(iv) Write another formula to show how far the faster is ahead of the slower when the latter has gone a certain distance (D_2, d_2).

6. Write down formulæ corresponding to (i), (iii), and (iv) of the preceding question supposing that the faster motor-car starts a certain distance ahead of the slower (d_0).

7. The "Lusitania" leaves New York to cross the Atlantic a certain number of sea-miles behind an ordinary liner. Given the number of knots (i.e. sea-miles per hour) at which each of the steamers is travelling, write formulæ to find:—

(i) How many miles nearer the steamers will be after a given number of hours than they were at starting (m, d_0, s_1, s_2, t);

(ii) What will then be their distance apart (d);

(iii) How many hours it will take the "Lusitania" to catch the other liner up (T);

(iv) What distance they will then be from New York (L);

(v) How many hours it will take to reduce their distance apart *by* a given number of miles (M);

(vi) How many hours it will take to reduce their distance apart *to* a given number of miles (D);

(vii) The distance of the "Lusitania" from New York; given the distance of the other steamer from New York (L_1, L_2);

(viii) The distance of the slower steamer from New York when the "Lusitania" is a given number of miles from New York (L_2, L_1);

(ix) The distance of the slower steamer from New York when the two steamers are a given distance apart (L_2, d);

(x) The distance of the "Lusitania" from New York when the two steamers are a given distance apart (L_1, d).

8. Write down a formula for :—

(i) The length of wall-paper of given width required to cover a wall of given length and height. All the dimensions being in feet (L, w, l, h) ;

(ii) The same, the width of the paper being given in inches ;

(iii) The same, the length of the paper being required in yards, but the dimensions of the room being still given in feet, and the width of the paper in inches ;

(iv) The same, the width of the paper being the standard width of 21 inches ;

(v) The number of "pieces" of standard width required for the wall, each piece having the standard length of 12 yards ;

(vi) The cost of the paper in shillings, given the price of the paper per piece in shillings (C, p) ;

(vii) The cost of the paper in shillings, the price of a piece being given in pence.

9. (i) Into a jug of uniform cross-section containing a certain amount of water, I drop a number of marbles of equal size. Write a formula for the height through which the surface of the water will rise, supposing that all the marbles are covered (h, A, n, v).

(Does the shape of the cross-section make any difference ?)

(ii) Given the original depth of the water, find the depth after a given number of marbles have been dropped in (d_n, d_o).

(iii) Write a formula for finding the number of marbles required to raise the surface of the water from one given level to another.

(iv) The Crow in Æsop's Fable drank water from a deep vessel by dropping in pebbles until the water was sufficiently high. Supposing the vessel to be a uniform jug of given cross-section, and the pebbles to be of equal size, and given the length of the crow's bill, and the original distance of the surface of the water from the top of the jug, find a formula for the number of pebbles that he must drop in so as to drink comfortably. (You should allow one more pebble than the least number necessary) (n, A, v, b, d).

10. Write down a formula for :—

(i) The weight of a cylinder, given the radius of the base, the height, and the weight of a cubic inch of the material (W, r, h, c) ;

(ii) The weight of a cylindrical bottle containing ink, given the weight of the empty bottle, its internal radius, the depth of the ink and the weight of a cubic inch of it (W, w, r, d, c) ;

(iii) The weight of a cubic inch of the ink, given the weight of the bottle, with the ink in it, the weight of the empty bottle, the internal radius of the bottle and the depth of the ink ;

(iv) The cost of the ink per cubic inch, given the price of the bottle when full, the internal radius, and the depth of the ink (C, P, r, d).

B. Substitution.

11. Where a railway line goes round a curve the outer rail is always raised above the inner rail.

$$h = \frac{wv^2}{1{\cdot}25r}$$

$h \equiv$ elevation of outer rail in inches; $w \equiv$ width of gauge in feet; $v \equiv$ greatest speed of train in miles per hour; $r \equiv$ radius of curve in feet.

(i) Find how much the outer rail must be raised round a curve of 2260 feet radius on an ordinary English "narrow gauge" railway (w = 4 feet $8\frac{1}{2}$ inches), the greatest speed allowed being 60 miles per hour.

(ii) Solve the same problem for a railway with the Irish gauge of 5 feet 3 inches, the radius being 2800 ft.

(iii) Light railways in India have a metre gauge (practically 3·3 feet). Find the elevation of the outer rail to allow a speed of 24 miles per hour round a curve of 264 yards radius.

12. A moving train experiences a resistance due to the air, friction of the rails, etc. Harding's formula gives

$$R = W\,(6 + 0{\cdot}33v) + 0{\cdot}0025Av^2$$

R $\equiv$ resistance in lbs. on the level in calm weather; W $\equiv$ wt. of train in tons; A $\equiv$ area of frontage of train in square feet.

(i) Calculate the resistance to a train weighing 30 tons, presenting to the air a frontage of 60 square feet, when it is going at 50 miles an hour;

(ii) The same, when the train weighs 24 tons and is going at 40 miles per hour.

13. The horse-power required to drive a ship at a given speed can be found (approximately) by means of the formula:—

$$H = 0{\cdot}0088s^3\,(0{\cdot}05A + 0{\cdot}005S)$$

$s \equiv$ speed in knots; A $\equiv$ immersed cross-section in square feet; S $\equiv$ area of wetted surface in square feet.

(i) What horse-power is needed to drive a steamer at a speed of 10 knots, when A = 200 square feet and S = 2000 square feet?

(ii) A steamer in which A = 240 square feet and S = 3000 square feet is travelling at 12 knots. What horse-power is being used?

14. The diameter in inches of a crank-shaft[1] (when made of wrought iron) is given by the formula :—

$$d = \sqrt[3]{\frac{83\mathrm{H}}{n}}$$

Find the thickness of crank-shaft that should be used in a 16 horse-power gas engine making 120 revolutions per minute.

15. If you look in at the door of an electric lighting works or a factory you can generally see near the roof a long shaft with large pulley wheels on it conveying the power from the engine to the dynamos or other machines by means of leather belts. A sufficient thickness for this shaft is given by :—

$$d = \sqrt[3]{\frac{65\mathrm{H}}{n}}$$

Find the diameter of a shaft which is made to revolve 180 times a minute and is transmitting 120 horse-power from an engine to a number of machines.

16. What information is given by the following formula?

$$\mathrm{B} = \frac{2880d^3}{\mathrm{D}^2 l}$$

$\mathrm{B} \equiv$ the pressure of steam in the boiler; $\mathrm{D} \equiv$ diameter of cylinder; $d \equiv$ diameter of crank-shaft; $l \equiv$ length of piston-stroke, all three in inches.

What should be the boiler pressure if the cylinder's diameter is 50 inches, the length of the piston-stroke 30 inches, and the shaft 8 inches in diameter?

17. There should also be a certain connexion between the diameter of the crank-shaft and the dimensions of the cylinder. The connexion is :—

$$\mathrm{D} = \sqrt{\frac{6{\cdot}55d^3}{l}}$$

What is the diameter of the cylinder suitable to a crank-shaft 12 inches in diameter, the length of the stroke being 3 feet?

18. The rate at which the draught rushes up a factory chimney is given by the formula :—

$$s = 2{\cdot}42\left(1 - \frac{t}{\mathrm{T}}\right)\sqrt{h}$$

[1] For Nos. 14-17 refer to fig. 4 and the accompanying note (p. 30).

$s \equiv$ speed in ft./sec.; $h \equiv$ height of chimney above fire-bars; $T \equiv$ internal temperature (degrees Fahrenheit); $t \equiv$ outside temperature.

(i) Calculate the draught in a chimney 169 feet high, the inside and outside temperatures being 360° and 60° respectively;

(ii) The same, the chimney being 289 feet high, the internal and external temperatures being respectively 580° and 50°.

19. It has been found by observations in Madras that the greatest number of cubic feet of water discharged per second in a time of flood is given by the formula:—

$$w = k \sqrt[3]{A^2}$$

$A \equiv$ no. of square miles drained by the river; $k \equiv$ a constant, which has the value 450 for places within 15 miles of the sea, 562·5 between 15 and 100 miles from the sea, 675 for limited areas near hills.

(i) Find the greatest flood discharge to be expected in a Madras river at a point 10 miles from the sea, the drainage area being 8000 square miles;

(ii) The same, substituting 60 miles and 3700 square miles;

(iii) The same, for a river in hill country draining 1200 square miles.

20. The following formulæ state the rules for determining the income-tax (in shillings) payable on *earned* incomes. (Incomes entirely or partly unearned are subject to different rules.)

(1) $160 < I \ngtr 400 : T = \frac{3}{4}(I - 10C - P - 160).$
(2) $400 < I \ngtr 500 : T = \frac{3}{4}(I - 10C - P - 150).$
(3) $500 < I \ngtr 600 : T = \frac{3}{4}(I - P - 120).$
(4) $600 < I \ngtr 700 : T = \frac{3}{4}(I - P - 70).$
(5) $700 < I \ngtr 2000 : T = \frac{3}{4}(I - P).$
(6) $2000 < I \ngtr 3000 : T = I - P.$
(7) $3000 < I \ngtr 5000 : T = \frac{7}{6}(I - P).$
(8) $I > 5000 : T = \frac{7}{6}(I - P) + \frac{1}{2}(I - 3000).$

($I \equiv$ earned income; $C \equiv$ no. of children under 16; $P \equiv$ amount of insurance premiums or superannuation fund payment.)

Explain the rules and apply them to calculate the tax payable:—

(i) By A, who earns £360 per annum, has 4 children, and pays £12 in insurance premiums on the lives of his wife and himself;

(ii) By B, who earns £475, has 5 children, and pays £16 to a superannuation fund and £10 in insurance premiums;

(iii) By C, who earns £660, has 3 children, and pays £70 in insurance premiums;
(iv) By D, who earns £1800;
(v) By E, who earns £2400, and pays £225 in insurance premiums;
(vi) By F, who earns £7600, and pays £500 in insurance premiums.

C. Some Arithmetical Puzzles.

21. I take two numbers, each less than 100. I multiply the first by 100 and add the second. The result is in various cases (i) 1426, (ii) 2853, (iii) 947, (iv) 4308, (v) 1008. What were the numbers in each case?

Note.—The principle of No. 21 is used in the following puzzles, Nos. 22-25.

22. Taking the number of the month in which I was born I multiply by 20, add 12, multiply the sum by 5, and then add the day of the month. The result is 578. When is my birthday? What is the rule for solving the puzzle?

23. I am thinking of a date in English history. I take 4 from the number which marks the century and multiply the residue by 10. I then add 4 to this product and again multiply by 10. Finally, I add the number of years in the century. The result is 1289. What was the date? What is the rule for solving the puzzle?

24. I am thinking of a number. To its double I add 4. I multiply the sum by 5 and add 3. I now multiply by 10 and add another 3. The result is 26733. What was the number? The rule?

25. I open a book at a certain page and choose one of the first 9 words in one of the first 9 lines. To 10 times the number of the page I add 25 together with the number of the line. To 10 times this sum I add the number of the word. The result is 12697. Tell me the numbers of the page, line, and word. What is the rule?

26. Invent puzzles similar to Nos. 22-25 but involving different calculations.

27. A number N is composed of two digits a and b. Show that

$$N = 9a + (a + b)$$

Hence prove (i) that if the sum of the digits is divisible by 9 the number itself is divisible by 9 and conversely; (ii) that

if the sum of the digits is divisible by 3 the number is divisible by 3 and conversely.

28. A number is composed of three digits a, b, c, and is divisible (i) by 9, (ii) by 3. Show that the sum of the digits is divisible (i) by 9, (ii) by 3.

29. Do the rules of Nos. 27 and 28 apply to a number of four or more digits?

30. Find which of the following numbers is divisible by 9, and which of the others are divisible by 3: 426, 891, 3024, 12765, 42936, 824315, 927654727, 824510682.

31. In order to find whether 726 is divisible by 11 I add the 7 to the 26. Since 33 is divisible by 11 I conclude that 726 is also divisible by that number. Show that this rule holds good for all numbers containing three digits.

32. To find whether 5379 it divisible by 11 I add 79 to 53 and obtain 132 which is divisible by 11. I conclude that 5379 is also a multiple of 11. Show that this rule is true for all numbers composed of four digits.

33. What are the corresponding tests for numbers composed (i) of five digits, (ii) of six digits? Apply them to the numbers 21736, 932085.

34. Show that any number of four figures, such as 8228, in which the second half is the first half reversed, must be devisible by 11.

35. Show that the same rule holds for numbers containing six figures such as 827728.

Note.—A number whose digits are a and b can be written $11a + (b - a)$ or $11a - (a - b)$ according as b or a is greater.

36. Show that if a number of three digits is divisible by 11 the difference between the middle digit and the sum of the other two must be zero or 11.

37. Show that if a number of four digits is divisible by 11 the sum of the first and third digits is either the same as the sum of the second and fourth or differs from it by 11. Can the rule be extended?

38. Test the divisibility by 11 of the following numbers: 102036, 151602, 9243871, 8654768.

39. (i) Factorize $(at + b) - (bt + a)$, a being greater than b.

(ii) Factorize $(at^2 + bt + c) - (ct^2 + bt + a)$, a being greater than c.

40. Use the results of No. 39 to prove the following: Take any number composed either of two or of three digits. Obtain another number by reversing the order of the digits. The difference between the numbers will be a multiple of nine.

41. Show that when there are three digits the middle digit of the difference is always 9. What must be the sum of the other two digits? (See No. 28.)

42. I take a number of three digits and form another number by reversing the digits. On subtracting I find that the first figure of the difference is 4. What are the other two figures?

43. In other similar cases the last figure of the difference was (i) 7, (ii), 9, (iii) 0. What were the figures in each case?

44. Express (i) $(at^3 + bt^2 + ct + d) - (dt^3 + ct^2 + bt + a)$ and (ii) $(at^4 + bt^3 + ct^2 + dt + e) - (et^4 + dt^3 + ct^2 + bt + a)$ each as the sum of two products.

45. Use No. 44 to show that the rule of No. 40 holds good for numbers containing four and five digits.

46. I write down a number containing five digits and a second number whose digits are those of the former taken in reverse order. I subtract the larger of these numbers from the smaller. I cross out one of the figures of the remainder. The other figures are 3, 0, 9, 7. What figure did I cross out?

47. I do the same with a number containing six figures. The other figures in the remainder are 1, 3, 6, 3, 5. Show that the figure crossed out must have been one of two.

48. Write down any sum of money (£a bs. cd.) less than £12, the numbers of pounds and pence being different. Obtain a new sum by reversing the order of the figures (£c bs. ad.). Find the difference between the two sums. Reverse the order of the figures of the difference and so obtain a fourth sum. Add together the third and fourth sums. Show that the result will always be £12 18s. 11d.

49. Show that $(5 + a)(5 + b) = 10(a + b) + (5 - a)(5 - b)$.

50. The following is an example of the *Regula Stultorum* (The Fool's Rule) used in the Middle Ages for multiplying two numbers between 5 and 10. Assign the numbers 6, 7, 8, 9, 10 in order to the fingers of each hand, counting the thumb as the first finger. To multiply 7 by 9 bring together the tips of the fingers which bear these numbers. Including these two themselves there are now six fingers to the front

of the point of contact. Therefore the first figure of the product is 6. Also there are behind the point of contact three fingers on one hand and one on the other. Therefore the second figure in the product is $3 \times 1 = 3$.

Use No. 49 to prove this rule.

D. Graphic Representation.

51. The following table gives the height of the spring tides at points on the River Thames, at specified distances from the mouth of the river. Draw a graph based upon the table. Use it to determine:—

(i) The height of the spring tides at Barnes Bridge, 70·5 miles from the mouth of the river;
(ii) The places where the height is 10 feet and 5 feet respectively;
(iii) How far from the mouth of the river the tidal influence would be felt if the locks did not intervene;
(iv) The probable height of the spring tide at Greenwich, 50 miles from the mouth.

Place.	*Distance.* Miles.	*Height.* ft. in.
London Docks	60	18 10
Putney	67·5	10 2
Kew	73	7 1
Richmond	76	3 10
Teddington	79	1 4½

52. The following table gives the times of sunrise and sunset in London on certain dates in May, June, and July. Construct from it a graph showing the length of the day for all dates between 1 May and 31 July. (The day may be taken to begin one hour before sunrise and to end an hour after sunset.) When is the longest day and how many hours does it last? During how many days in the year is the night not more than six hours long? (Read the note above, Ex. II, No. 17.)

Day.	*May.* Rises.	Sets.	*June.* Rises.	Sets.	*July.* Rises.	Sets.
10	4.20	7.32	3.45	8.13	3.56	8.14
20	4.4	7.48	3.44	8.18	4.7	8.5
30	3 52	8.2	3.48	8.18	4.21	7.51

53. The following table gives the temperature at different times of a quantity of mercury which had been heated and was

then allowed to cool. Draw a graph representing the probable history of its cooling. Why do you think that this history is more likely to be described by the curve you draw than by any other?

Time in min.:	0	1	2	3	4	5	6	7	8
Temperature (F.)	425°	296°	230°	188°	160°	136°	120°	108°	100°

54. Draw a horizontal "time-line" on the same scale as the one in No. 53. Midway between the divisions 0-1, 1-2, 2-3, etc., erect perpendiculars to represent the fall of temperature of the mercury during the first, second, third, etc., minutes. Through the tops of these perpendiculars draw a smooth curve. Deduce from this curve how much the mercury cooled (i) between $1\frac{1}{2}$ and $2\frac{1}{2}$ minutes, (ii) between 3·2 and 4·2 minutes after the beginning of the observations. In which minute did it cool 20°? Confirm these deductions by means of the graph of No. 53.

55. Booth's "Life and Labours of the People in London" gives the following particulars about the number of men and boys (per 10,000 of the male population whose ages are between 10 and 80) who are employed between different ages. Exhibit the information in a column-graph. Draw a continuous curve to show the probable distribution of employment at the intervening ages. How many persons are employed between (i) 14 and 18, (ii) 40 and 45, (iii) 45 and 50, (iv) 60 and 65?

Ages.	*Number Employed.*
10 and under 15	193
15 ,, 20	880
20 ,, 25	933
25 ,, 35	1636
35 ,, 45	1201
45 ,, 55	830
55 ,, 65	434
65 ,, 80	192

56. A small dog is swimming across a pond along the path ABC (fig. 26) with a stick in his mouth. When he is at B a larger dog jumps into the pond at D in order to capture the stick. The big dog swims from moment to moment directly towards the other. While the smaller dog swims a distance P the larger swims a distance Q. Trace the path of the larger dog and find whether he will overtake the smaller one before he lands at C. (The figure may be traced through thin paper or by holding it against a window.)

57. Draw a long rectangle to represent the complete value of $1/(1 - a)$ when $a = \frac{1}{5}$. Indicate the lengths which represent the first and second approximations to the value of the fraction.

Do the same with regard to $1/(1 + a)$.

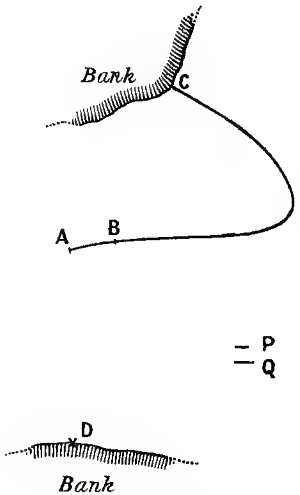


FIG. 26.

58. A square measuring 10 cms. each way is made into one measuring $(10 + h)$ cms. each way by the addition of a "gnomon" consisting of two equal rectangles, R and R′ and a square Q. Draw a series of strips representing the area of the gnomon in the cases when h is 1 cm., 0·8 cm., 0·6 cm., 0·4 cm., and 0·2 cm. respectively. (The width of the strip should be the value of h in each case.) Shade the rectangles and leave the square Q unshaded so as to indicate the amount of the error committed in each case by taking the increase in the original square as equal to 20 h.

59. A cube measuring 10 cms. each way is made into one measuring $(10 + h)$ cms. by the addition of solids to three of its faces as in Ex. XI. Draw a long strip whose *area* shall represent the total *volume* of the solids added when h has the values 1, 0·8, 0·6, 0·4, 0·2. (The height should be the value of 10 h.) Mark off lengths representing the volumes which involve h, h^2, and h^3 respectively, so as to indicate the relative importance of their contributions to the whole.

60. The bearings of an aeroplane were taken every minute at two stations, A and B, situated 2000 yards apart on an east and west line. A record of the height of the aeroplane was kept by a passenger. The observations are given below. Draw a "bird's-eye view" of the track of the aeroplane. By measuring the length of the line drawn determine the horizontal distance covered by the flight of the aeroplane. On a horizontal line of the proper length mark verticals representing the recorded heights of the aeroplane. Draw a smooth curve

showing the probable height at any other time. Cut along the curve and the base line. Fix the strip so obtained along the plan of the track so that the curve may represent the actual path of the aeroplane through the air. What was the average horizontal speed? When did the aeroplane pass between the two stations, at what distance from each, and at what height? When was the aeroplane nearest to A and to B? What were its distances from A and B (measured horizontally) and what was its height at each of these moments?

Minute.	*Height above Ground in feet.*	*Bearing from A.*	*Bearing from B.*
0	0	due N.	76° W. of N.
1	120	$31\frac{1}{2}$° E. of N.	79° W. of N.
2	390	$83\frac{1}{2}$° E. of N.	88° W. of N.
3	420	$76\frac{1}{2}$° E. of S.	75° W. of S.
4	570	$75\frac{1}{2}$° E. of S.	$42\frac{1}{2}$° W. of S.
5	360	81° E. of S.	38° E. of S.
6	180	89° E. of N.	86° E. of N.
7	270	$77\frac{1}{2}$° E. of N.	22° E. of N.
8	480	$67\frac{1}{2}$° E. of N.	33° W. of N.
9	480	54° E. of N.	$53\frac{1}{2}$° W. of N.
10	300	35° E. of N.	$64\frac{1}{2}$° W. of N.
11	0	10° E. of N.	71° W. of N.

E. Factorization, etc.

Write formulæ in the form most convenient for calculating the numbers specified in Nos. 61-68. (The linear measurements indicated in the figures may be taken to be centimetres.)

61. The volume (V) of a block of wood of height h and of uniform cross-section represented by fig. 27.

62. The density (d) of the wood if the block of No. 61 weighs W grams.

63. The volume (V) of a block a by a by $3b$ with a hemisphere of radius b scooped out of the top and bottom faces. (Fig. 28 would be a section across the middle of this block.)

64. The volume of the figure obtained by rotating fig. 27 about its axis PQ. (This figure can be thought of as representing a boiler with hemispherical ends.)

65. The volume of the figure obtained by rotating fig. 28 about the line PQ.

66. The volume (V) of a column of uniform cross-section represented by fig. 28 and of height h.

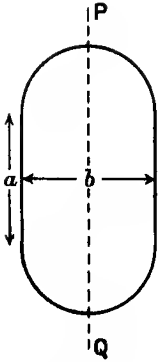


FIG. 27.

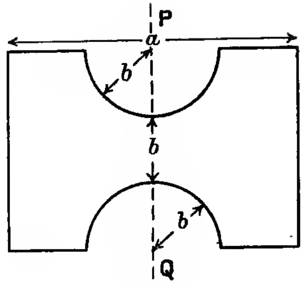


FIG. 28.

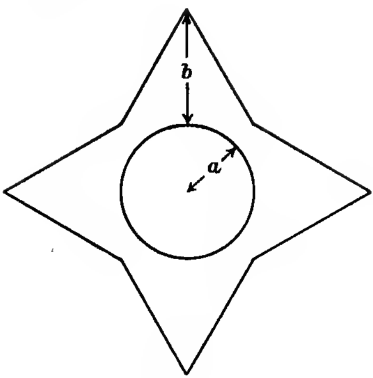


FIG. 29.

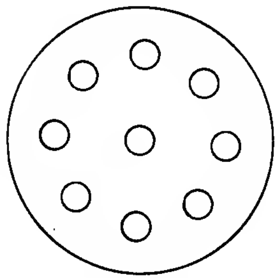


FIG. 30.

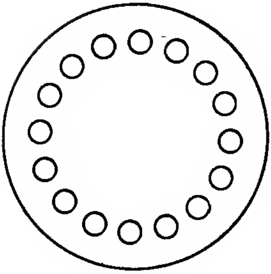


FIG. 31.

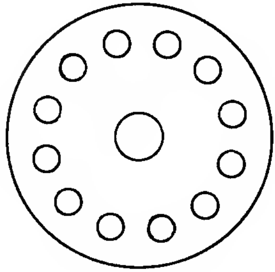


FIG. 32.

67. The weight (W) of the column of No. 66 if the material weighs w grams/cm.^{3}

68. Write down formulæ expressed in the form most suitable for calculation which shall give the total surface of—

(i) a cylinder ; (ii) a four-sided pyramid on a square base ; (iii) a cone. [Use the following symbols : (i) r, h ; (ii) s, l ; (iii) r, l.]

69. Write down a formula to give in the most convenient form the surface area of the boiler of No. 64.

70. Write a formula for the area of a plate like fig. 29, pierced with a circular hole.

71. Figs. 30 and 31 represent discs pierced respectively with nine and sixteen equal holes. Find expressions for the area of the surface of each (A, R, r).

72. Fig. 32 represents a disc pierced by thirteen circular holes, the radius of the large hole in the middle being double the radius of the smaller holes. If the radius of the disc is R and that of one of the small holes r, find the area of the surface of the disc.

73. Find the formulæ for

(i) The area of the upper surface of a plate of radius a pierced with any square number (n^2) of holes of radius b ;
(ii) The weight of such a plate given its thickness (c) and the weight (w) of a cubic unit of the material.

Why does the question concern only *square* numbers of holes ?

74. Find formulæ for the area of fig. 24 (p. 41), (i) calling the radius of the hole a and the middle of the ring b ; (ii) calling the width of the ring w and the radius of the hole r ; (iii) calling the outer radius of the ring r and the width w.

75. Find formulæ

(i) For the total internal and external curved surface of a pipe whose external and internal radii are R and r and whose length is l ;
(ii) For the total surface of such a pipe counting in the top and bottom ring-surfaces.

76. Find two formulæ corresponding to those of No. 75, given the internal radius (r) and the thickness (t) of the pipe.

77. Find two formulæ corresponding to those of No. 75, given the external radius (r) and the thickness of the pipe.

78. (i) The top of a ladder 15 feet long rests against a wall at a point 12 feet above the ground. How far is the foot of the ladder from the wall?

(ii) Write down a formula for finding the distance (d) of the foot of the ladder from the wall, given the length of the ladder (l) and the height (h) of the point against which it rests.

79. A kite at the end of a string l feet long is immediately over the head of a boy who is standing d feet from the boy who is flying it. Give a formula for calculating the greatest possible height of the kite above the level of the latter boy's hand.

Why are you not asked to give a formula for the actual height of the kite?

80. After walking due east for t hours at the rate of 3 miles an hour I find I am due south of a tower on a hill d miles from my starting-point. How shall I calculate my distance from the tower?

81. Each of the following expressions describes the factors of a product, a being a symbol for any number you please. Write the expressions which describe the products.

(i) $(a + 3)(a + 2)$. (ii) $(a + 3)(a - 2)$.
(iii) $(a - 8)(a - 5)$. (iv) $(a - 9)(a + 4)$.
(v) $(a + 3)(a + 3)$. (vi) $(a - 7)^2$.
(vii) $(2a + 1)(3a + 2)$. (viii) $(4a - 3)(3a - 4)$.
(ix) $(7a - 10)(3a + 5)$. (x) $(7a - 10)(3a + 4)$.
(xi) $(15a - 7)(12a + 11)$. (xii) $(2a + 3)^2$.
(xiii) $(5a - 6)^2$. (xiv) $(2a^2 - 5)(5a^2 + 2)$.

Verify your answers to (iv), (viii), (ix), and (xiv) by substituting for a any numbers you please.

82. In the following expressions a and b are symbols for any numbers. Write symbolic expressions of the products of the factors described. Test the accuracy of the descriptions in any three cases you please by substituting numbers for a and b.

(i) $(a + 2b)(a - 7b)$. (ii) $(a - 4b)(2a - b)$.
(iii) $(5a + 2b)(2a + 5b)$. (iv) $(7a - 4b)^2$.
(v) $(10a - 4b)(7a + 3b)$. (vi) $(13a - 5b)(2a + 7b)$.
(vii) $(a + 2b)^3$. (viii) $(2a - b)^3$.
(ix) $(2a - 3b)^3$. (x) $(4a + 3b)^3$.

83. Complete the following identities:—

(i) $(a^2 + 5a + 4)/(a + 1) =$.
(ii) $(a^2 - 5a + 4)/(a - 4) =$.
(iii) $(a^2 + 3a - 4)/(a - 1) =$.

(iv) $(a^2 - 2a - 10)/(a - 5) = \quad .$
(v) $(4a^2 + 8a + 3)/(2a + 1) =$
(vi) $(35a^2 + 13a - 12)/(7a - 3) = \quad .$
(vii) $(6a^2 - 5ab - 6b^2)/(3a + 2b) = \quad .$
(viii) $(2a^2 - 11ab + 15b^2)/(a - 3b) = \quad .$
(ix) $(6a^2 - ab - 35b^2)/(2a - 5b) = \quad .$
(x) $(49a^2 - 28ab + 4b^2)/(7a - 2b) = \quad .$

84. Factorize the following expressions:—

(i) $a^2 \quad 5a + 6.$
(ii) $p^2 - 5p - 6.$
(iii) $p^3 + 5p - 6.$
(iv) $a^2 - a - 132.$
(v) $a^2 - 6a + 9.$
(vi) $p^2 - 14p + 49.$
(vii) $a^2 - 3ab + 4b^2.$
(viii) $4a^2 + 3ab + b^2.$
(ix) $4a^2 + 12ab + 9b^2.$
(x) $6a^2 - 13ab - 5b^2.$
(xi) $2a^2 + 5ab - 12b^2.$
(xii) $55a^2 + 7ab - 6b^2.$
(xiii) $ab + 3a - b - 3.$
(xiv) $ab - 3ap + bq - 3pq.$
(xv) $2ap - 4aq + 3bp - 6bq.$

85. Complete the identity $(a + b)(c + d) = \quad .$ Illustrate it by a diagram showing how from four adjacent rooms with certain dimensions a large room $(a + b)$ long and $(c + d)$ wide can be formed.

86. Complete the identity $(a - b)(c - d) = \quad .$ Illustrate it by taking a thin cardboard or paper rectangle, a long and c wide, and reducing it, by certain subtractions and additions, to a rectangle $(a - b)$ long and $(c - d)$ wide.

Note.—The expression $a^2 + 2ab$ describes a figure like fig. 19, in which AF $= a =$ CD and FE $= b =$ ED. The figure may be made into the square $(a + b)^2$ by filling in the square FD $= b^2$. The addition of b^2 to $a^2 + 2ab$ is said to **complete the square.** The complete expression $a^2 + 2ab + b^2$ is called a **perfect square.**

Next let AB $= a - b$ and FE $= b$. Produce BA to C and BC to H, making AG $=$ CH $= b$ and so obtain the square a^2. In this case fig. 19 will be represented by $a^2 - 2ab$, and to complete the square $(a - b)^2$ it is again necessary to add b^2.

87. What additions (or subtractions) are necessary to make the following expressions perfect squares (or multiples of perfect squares)?

(i) $a^2 + 6a.$
(ii) $a^2 - 8a.$
(iii) $a^2 + 12a.$
(iv) $a^2 - 22a.$
(v) $a^2 + 3a.$
(vi) $a^2 - 7a.$
(vii) $a^2 + 14a + 20.$
(viii) $a^2 - 8a - 3.$
(ix) $p^2 + 13p + 1.$
(x) $p^2 - 10p + 42.$
(xi) $a^2 - 4ab.$
(xii) $a^2 + 6ab + 20b^2.$
(xiii) $9a^2 + 12ab.$
(xiv) $16p^2 + 40pq.$

(xv) $4m^2 - 16mn + 7n^2$. (xvi) $3a^2 + 12ab$.
(xvii) $5s^2 - 30sh - 7h^2$. (xviii) $6p^2 - 12p + 17$.
(xix) $3a^2 + 15ab$. (xx) $2a^2 - 7ab + b^2$.

88. In the following equivalences $s \equiv$ "the side of a certain square". Find (by completing the square) what is the length of the side in each case.

(i) $s^2 + 2s = 8$. (ii) $s^2 + 12s = 13$.
(iii) $s^2 - 12s = 13$. (iv) $s^2 + 8s + 65$.
(v) $60 - 14s + s^2 = 36$. (vi) $s^2 - 3s = 13{\cdot}75$.
(vii) $s^2 - 2a = 3a^2$. (viii) $s^2 + 6a = 7a^2$.
(ix) $s^2 - 18a + 27 = 46$. (x) $s^2 - \frac{7}{2}s + \frac{1}{2} = 5$.

89. Let the volume of the complete cube model (Ex. XI) be a^3 and that of the inner cube b^3; then the additions have a volume $a^3 - b^3$. Lay these additions side by side on the table so as to show that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Prove by multiplication that the result holds good for all numbers.

90. Call the volume of the inner cube a^3 and that of the small cube b^3. Stand the latter upon the former and note what further additions would make a column having a^2 for its base and $(a + b)$ for its height. Hence show that

$$\begin{aligned} a^3 + b^3 &= (a + b)\,a^2 - (a^2 - b^2)\,b \\ &= (a + b)(a^2 - ab + b^2) \end{aligned}$$

Prove the universal validity of the identity.

F. Approximations.

91. Find the difference between the distances of the horizons visible from the bottom and top of a tower 60 feet high built on the side of a hill 666 feet above the sea ($\sqrt{74} = 8{\cdot}6$). (See Ex. IX, No. 29.)

92. A pendulum consists of a small heavy ball at the end of a light thread of length l. When wetted the thread stretches by a small amount h. Write formulæ for calculating approximately (i) the new time of swing of the pendulum (t'), and (ii) the increase (i) in the time of swing. (See Ex. VII, No. 11.)

93. Draw a straight line, AB, of length a. At B erect a perpendicular of length b, small compared with a. Write down a formula for calculating the hypothenuse c to a first approximation.

94. Verify the formula of No. 93 either by drawing a figure on a large sheet of paper or by arranging pins and strings on the floor or a large table. Choose your own values for a and b.

95. The centre of the bob of a pendulum is 30 inches below the nail to which it is attached and is being held 6 inches out of the vertical. Calculate the approximate length of the pendulum from nail to centre of bob.

96. As measured on a map the summit of a hill is 3 miles from a certain house at its foot. Between these points the difference of level is $\frac{1}{4}$ of a mile. By how many yards does the direct distance between the house and the summit exceed the horizontal distance?

97. A boy on a cliff 325 feet above the sea descends a distance of 40 feet. What difference does the descent make to the distance he can see? ($\sqrt{13} = 3{\cdot}61$.) Prove the formula you use.

98. What difference is made to the time of swing of a pendulum 4 feet long if it is shortened 3 inches? Prove your formula.

99. The radius of the earth may be taken as 4000 miles and was at one time rather greater. Assuming that it was once 10 miles greater than at present, calculate approximately (i) the superficial area, (ii) the volume which it has lost. (The formulæ for the area and volume of a sphere are

$$A = 4\pi r^2 \text{ and } V = \tfrac{4}{3}\pi r^3).$$

100. A quantity a is increased by a small amount h and it is desired to know the approximate increase in a^2 and a^3. Show that the error in taking the increase in a^2 to be $2ah$ is not more than $h/2a$ of the whole increase, and the error in taking the increase in a^3 to be $3a^2h$ not more than $(h/a + h^2/3a^2)$ of the whole increase. Apply these results to the calculations of No. 99.

G. Changing the Subject, etc.

101. Take the formula of No. 1 (ii) and change the subject to m. What information does the new formula give? Is it correct?

102. Change the subject of the formula of No. 2 (ii) to b. What does the new formula describe?

103. In No. 5 change the subject of (iii) to s_1 and of (iv) to s_2.

104. In No. 7 change the subject of (ix) to s_2 and of (x) to s_1.

105. Deduce a formula for v from the one given in No. 11. Use it to find the greatest speed allowable when a train goes round a curve of 2000 feet radius, the gauge being 5 feet and the outer rail raised 9 inches above the inner rail.

106. Change the subject of No. 15 to H. Calculate the horse power that may be transmitted safely by a 4-inch shaft revolving 195 times a minute.

107. Change the subject of No. 16 (i) to D, (ii) to d. Interpret the new formulæ.

108. Change the subject of No. 13 to s. Calculate the speed at which the steamer in No. 13 (i) would travel if the engines were working at 220 horse-power. [$\sqrt[3]{10} = 2{\cdot}154$.]

109. A square lawn measures s feet each way. The owner wishes to enlarge it into a square of area A square feet by adding a feet to each side. Change the formula $A = (s + a)^2$ into one for calculating a, given s and A.

110. A flower bed shaped like fig. 14 (p. 35) is made by adding triangles in which $b = 6$ to the sides of a square. Find the length a so that the bed may have a total area of 64 square feet. (See Note to No. 87.)

111. Give a general formula for calculating a when b and A (the total area) are given.

112. In the case of fig. 15 obtain a formula for calculating a when b and A are given.

113. By the addition of a feet to both its length and its breadth a lawn 30 feet by 20 feet becomes 875 square feet in area. Calculate a.

114. Obtain a general formula for calculating a when the original length and breadth and the final area of the lawn are given.

115. A lawn 40 feet by 30 feet is to be reduced to 1000 square feet of turf by cutting a strip from the width and a strip twice as wide from the length. Calculate to the nearest tenth of a foot the final dimensions.

116. In fig. 27 (p. 94) a is 11 cms. and the area 88 cms.^{2} Calculate b to the nearest millimetre.

117. Give a general formula for solving problems like No. 116 given a and A.

Note.—The numbers $n - 10$ and $10 - n$ yield on squaring identical results, $n^2 - 20n + 100$ or $100 - 2n + n^2$. If one

of the last two expressions is given it is impossible, therefore, to say whether it is the square of a number greater or less than 10. That can be told only from the context. Thus $n^2 - 20n = 44$ gives, on completion of the square,

$$n^2 - 20n + 100 = 144 \quad . \quad . \quad . \quad \text{(i)}$$
$$n - 10 = 12$$
$$n = 22$$

It is evident here that $n - 10$ is the required square root, for $10 - n = 12$ would be impossible. On the other hand, if we are told that

$$n^2 - 20n + 100 = 36 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

both $\qquad n - 10 = 6 \text{ and } 10 - n = 6$

or $\qquad n = 16 \text{ and } n = 4$

are possible.

Unless, therefore, the statement of the problem shows (as in Nos. 110-117) that it has only one answer, two must be sought. Can you tell (by considering lines (i) and (ii) above) when there will be two and when only one is possible?

118. Calculate (to two decimal places) the numbers which comply with the following conditions:—

(i) $n^2 - 8n + 7 = 0$. (ii) $n^2 - 12n + 4 = 49$.
(iii) $3n^2 + 12n = 180$. (iv) $2n^2 - 7n = 3$.
(v) $2{\cdot}2n^2 + 11n = 20$. (vi) $n^2 - 14n = 32$.
(vii) $n^2 - 20n + 91 = 0$. (viii) $150 - 52n + n^2 = 6$.
(ix) $(3n - 4)(2n - 1) = 20$. (x) $(2n - 1)(3n \quad 2) = 40$.

119. Calculate the numbers which comply with the following conditions:—

(i) $n(n + 1) = (n - 1)(n + 2) + 2$.

(ii) $3(n + 2)(n - 8) - (n - 4)(n - 6) = 2(n - 2)(n - 7)$.

(iii) $\dfrac{1}{n} + \dfrac{2}{n - 1} = \dfrac{3}{n - 2}$.

(iv) $\dfrac{n - 5}{5} = \dfrac{n - 4}{10} + \dfrac{n - 3}{40}$.

(v) $\dfrac{2}{3}n - 5 = \dfrac{5}{12}n + \dfrac{1}{4}$.

(vi) $\dfrac{3n - 1}{7} + \dfrac{6 - n}{4} - \dfrac{n - 2}{6} = 2 - \dfrac{n + 2}{28}$.

(vii) $3(n - 1) + 5(n - 2) + \dfrac{7n + 8}{3} = 0$.

(viii) $\dfrac{7n - 4}{15} - 2 + \dfrac{3n + 3}{16} = \dfrac{7n + 1}{20}$.

(ix) $$\frac{\frac{1}{n} - 3}{\frac{1}{n} - 2} = \frac{1}{4}.$$

(x) $$\frac{1}{2}\left(\frac{1}{n} - \frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{n} - \frac{1}{3}\right) = \frac{5}{6}\left(\frac{1}{n} - \frac{1}{5}\right).$$

120. Trace the steps by which you would determine the value of the numbers, which comply with conditions of the following form :—

(i) $a^2n - c = b^2n + d.$

(ii) $$\frac{n-a}{b} + \frac{n-b}{a} = 2.$$

(iii) $$\frac{1}{n-p} + \frac{1}{n-q} = \frac{2}{n}.$$

(iv) $$\frac{a}{n-a} - \frac{b}{n-b} = \frac{a-b}{n}.$$

(v) $$\frac{ac}{bn} - \frac{a}{2} = \frac{bc}{an} + \frac{b}{2}.$$

(vi) $$\frac{n+p}{q} + \frac{n-q}{p} = \frac{p+q}{q}.$$

(vii) $$\frac{n}{n-q} + \frac{p-q}{n} = \frac{n+p+q}{n+q}.$$

(viii) $a(2n - a) + b(2n - b) = 2ab.$

(ix) $(n + a)(n - b) + (n + a)b = n^2 + b^2.$

(x) $$\frac{\frac{1}{n} - a}{\frac{1}{n} - b} = \frac{b}{a}.$$

EXERCISE XVII.

DIRECT PROPORTION.

A.

1. Construct a ready reckoner for the cost in shillings (C) of given lengths (L) of picture moulding of three patterns. The first costs 4d., the second 9d., the third 1s. 3d. a foot.

Give the formulæ which correspond to the three graphs.

Find by the graphs how many feet of the first and second kinds of moulding cost as much as 16 feet of the third kind. Test your answers by the formulæ.

2. Draw (on one sheet) the graphs of the formulæ (i) $y = \frac{1}{4}x$, (ii) $y = 1{\cdot}2x$, (iii) $y = 2{\cdot}5x$. Find from the graphs the values of y corresponding to $x = 12$ in (i), $x = 4$ in (ii), $x = 2{\cdot}4$ in (iii). Compare the results with those obtainable from the formulæ.

3. A piece of cardboard containing 16 sq. cms., cut out of a uniform sheet, weighs 8·4 grms. Draw a graph giving the weight of pieces of the card of given area.

How should the graph be held so as to be a graph showing the area of pieces of the card of given weight?

Write formulæ giving (i) the weight (W) in terms of the area (A), and (ii) the area in terms of the weight.

4. It is found that 12 sq. cms. of another uniform sheet of cardboard weigh 7·8 grms. Write formulæ (i) for the weight of a given area, and (ii) for the area of a given weight.

5. The speed of a marble allowed to roll down a smooth slope is proportional to the time it has been rolling. In a given instance the speed was 12·5 feet per second after the marble had been rolling for 5 seconds. Write a formula giving the speed (v) in terms of the time (t). Calculate the speed after 12 seconds.

6. Two variables are in direct proportion. When the independent variable has the value 14·8 the value of the dependent variable is 3·7. Write the formula connecting them.

7. Two variables are connected by a relation of the form $y = kx$, and, when $x = 7{\cdot}2$, $y = 10{\cdot}8$. Find the value of k.

What would the relation be if the independent variable became the dependent variable, and *vice versa?*

8. Write a formula showing that W, the weight of a quantity of liquid, is directly proportional to V, the volume of the liquid.

Given that 10 cubic feet of water weigh 625 lb., adapt the formula for calculating the weight of given quantities of water.

9. Adapt the same formula for calculating the weights of given quantities of sea water, given that any quantity of sea water weighs 1·025 times as much as an equal quantity of fresh water.

B.

10. A bath already contains 24 gals. of water when a tap delivering $1\frac{1}{2}$ gals. a minute is turned on. Draw a graph showing the number of gallons in the bath at subsequent times. Also write the formula corresponding to the graph (Q, t).

Change the subject of the formula to t. How should the graph be held so as to correspond to the changed formula?

11. A bath had in it 24 gals. of water when the waste pipe was partially opened. It then began to empty at a constant rate of $\frac{3}{4}$ gal./min. Draw (on the sheet used for No. 10) a graph showing the amount of water left in the bath at given subsequent times. Write the formula corresponding to the graph.

Change the subject of the formula to t. Use the new formula to find when the bath will be empty. Compare the answer with that obtained from the graph.

12. Write down formulæ corresponding to the straight lines of Nos. 10 and 11, but not referring to any particular variables.

What do these formulæ become when the dependent variable replaces the independent variable, and *vice versa?*

13. The increase in the length of a vertical rubber cord is directly proportional to the weight hung at the end of it. When there is no weight the length is 16 inches. When a weight of 8 oz. is added the length becomes 17·2 inches.

Write a formula (i) for the increase in length in inches (i) due to a given weight in pounds (W), and (ii) for the total length of the cord (L) when supporting a given weight.

Change the subject of the second formula to W. What is the use of the new formula?

14. Draw (on one sheet) the graphs corresponding to the formulæ (i) $y = 20 + 3x$; (ii) $y = 20 - 3x$; (iii) $y = 4{\cdot}7 + 5{\cdot}3x$.

15. Water runs into an empty bath at the rate of 3 gals./min. After seven minutes the tap is turned on further and the rate of flow increases to 5 gals./min. Draw a graph showing the amount of water in the bath at various times.

Write a formula for the amount of water at a given time (i) during the first seven minutes, (ii) after the first seven minutes; the time being measured in each case from the moment when the tap was first turned on. Calculate the amount at the end of thirteen minutes. Compare the result with that obtained from the graph.

16. A motor runs from a point A at a constant speed of 24 mls./hr. for fifteen minutes when its speed is suddenly reduced to 17 mls./hr. Write formulæ for the distance it has covered in a given number of minutes since it passed A. Calculate its position (i) after ten minutes, (ii) after twenty-seven minutes. Compare the results with those obtained from a graph.

17. Change the subject of the second formula of No. 16 to t. Use the new formula to find when the car will have travelled 23 miles from A.

18. Water is running into a bath at a constant rate. After three minutes the bath contains 20 gals., after eight minutes 30 gals. Draw a graph showing the amount of water in the bath at different times. How would you calculate (i) the amount of water run in every minute (r), (ii) the quantity originally in the bath (Q_o)? Give the formula corresponding to the graph.

C.

Note.—When the graph showing the connexion between two variables is a straight line there is said to be a **linear relation** between them. What is the difference between saying this and saying that they are directly proportional?

19. Two variables x and y are connected by a linear relation. When $x = 8$, $y = 17$; when $x = 18$, $y = 30$. Find the formula connecting them.

20. The same, given that when $x = 7$, $y = 43$; and when $x = 20$, $y = 4$.

21. A motor-car after running for some time at 18 mls./hr. suddenly changes its speed to 24 mls./hr. Five minutes after the change it has travelled altogether 8 miles. Find (i) by a formula, (ii) by a graph, how long it ran at the lower speed.

22. In No. 18 the rate of flow is at a certain moment suddenly changed. After the water has been flowing for seventeen minutes altogether the bath contains 52 gals.; at the end of the twenty-third minute it contains 70 gals. Determine (i) by a graph, (ii) by a formula, the moment of the change. Give formulæ for the quantity of water in the bath at subsequent times, counting the time (iii) from the moment of change, (iv) from the beginning.

23. At the end of an entertainment the people leave a hall in a uniform stream. After one minute there are 1800 people left in the hall; after three minutes there are 1440. Shortly after this another door is thrown open with the consequence that seven and a half minutes after the performance there are only 360 people left, while at the end of eight and a quarter minutes altogether the hall is empty. Calculate the number of persons present at the end of the entertainment, the number who left per minute through each of the two doors, and the time when the second door was opened. Also write formulæ for the number of people in the hall at different times.

How many people were in the hall when the second door was opened? When was the number exactly 600?

24. Two variables, x and y, are connected by a linear relation. When x reaches a certain value they become connected by a different linear relation. When $x = 4$, $y = 37$; when $x = 7$, $y = 61$; when $x = 14$, $y = 65$; when $x = 20$, $y = 35$. At what value of x does the relation change? Find the formulæ of the two linear relations. For what value of x does $y = 0$? Confirm by drawing a graph.

TABLE OF TANGENTS.

To be read thus : Tan 28° = 0·532, etc.

Tan	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°
	0·00	0·017	0·035	0·052	0·070	0·087	0·105	0·123	0·141	0·158
Tan	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°
	0·176	0·194	0·213	0·231	0·249	0·268	0·287	0·306	0·325	0·344
Tan	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°
	0·364	0·384	0·404	0·424	0·445	0·466	0·488	0·510	0·532	0·554
Tan	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°
	0·577	0·601	0·625	0·649	0·675	0·700	0·727	0·754	0·781	0·810
Tan	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°
	0·839	0·869	0·900	0·933	0·966	1·000	1·036	1·072	1·111	1·150
Tan	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°
	1·192	1·235	1·280	1·327	1·376	1·428	1·483	1·540	1·600	1·664
Tan	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°
	1·732	1·804	1·881	1·963	2·050	2·145	2·246	2·356	2·475	2·605
Tan	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°
	2·747	2·904	3·078	3·271	3·487	3·732	4·011	4·331	4·705	5·145
Tan	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°
	5·671	6·314	7·115	8·144	9·514	11·43	14·30	19·08	28·64	57·29

EXERCISE XVIII.

THE USE OF THE TANGENT-TABLE.

Note.—Distances should be calculated to the nearest tenth of the unit, angles to the nearest half-degree. Neat diagrams should accompany solutions, but need not be drawn to scale.

A.

1. At a point (on level ground) 120 feet from the foot of a fir-tree the elevation of the summit of the tree is 35°. The observer's eye is 5 feet above the ground. How high is the tree?

2. What, to the same observer, would be the angle of elevation of the summit of the tree at a point (i) 100 feet, (ii) 200 feet from its foot?

3. At what distance from the tree would the angle of elevation be 61°?

4. Early this morning the shadow of an upright metre rule was 205 cms. long. What was then the angle of elevation (or "altitude") of the sun?

5. What would be the length of the shadow when the sun's altitude is (i) twice, (ii) three times as great as in No. 4?

6. Find the altitude of the sun when the shadow is (i) one-half, (ii) one-third of its original length in No. 4.

7. From an upper window in a house 160 feet from a church tower the angle of elevation of the top of the tower is 41° and the angle of depression of the bottom 15°. How high is the tower? How high is the point of observation?

8. Standing 60 feet away from a house I find that the altitude of a window-sill on the first floor is 27° and the altitude of the top of the wall 38°. How far is the window-sill below the roof?

9. Lying on a cliff 425 feet above sea-level I observed two boats both due west of me. The angle of depression of the

more distant was 12°, that of the nearer 18°. How far were they apart?

10. From the battlement of a castle tower 200 feet high I note that the line from my eye to the foot of the gateway makes an angle of 65° with the vertical face of the tower. How far is the gateway from the central tower?

11. I note also that the angle between the face of the tower and the line from my eye to the *top* of the gateway is 68°. How many feet is the top of the gateway below the level of my eye? How high is the gateway?

12. From the top of a vertical cliff I observe that the angle of depression of the summit of a lighthouse is 33° and that of the foot of the lighthouse is 37°. The lighthouse is 320 feet from the cliff. What is its height? How many feet is my eye above its summit?

13. A flagstaff stands on the roof of a building 115 feet high. Standing in the street some distance from the building I observe that the elevation of the bottom of the flagstaff is 37° and that of the top 46°. What is the height of the flagstaff? (My eye is 5 feet above the ground.)

B.

14. AB is a straight line of length l. From a point O the perpendicular OP falls between A and B and is of length p. The distances AP and BP are a and b respectively. The angle AOP is α and the angle BOP β. Find formulæ for a, b, and l in terms of p, α, and β.

15. Change the subject of the last formula of No. 14 to p.

16. Find a formula for p when P falls on AB produced beyond B.

17. AB and CD are two vertical lines of length H and h respectively, H being the longer. The straight line BD joining their bases is horizontal. The angles BAC and BAD are α and β respectively. Show that

$$h = \mathrm{H}\ (\tan\alpha - \tan\beta)/\tan\alpha.$$

(Consider how you did Nos. 11 and 12.)

18. Change the subject of the last formula to H. Make up a problem which could be solved by this formula.

19. ABC and CD are two straight lines at right angles. Let AB = d, CD = h, and let the angles ADC and BDC be α and β respectively. Find an expression for calculating d,

(Consider No. 9.) Change the subject to h. To what problem does the second formula correspond?

20. In the figure of No. 19 let DBC $= \alpha$ and DAC $= \beta$. Show that $d = h\ (\tan\alpha - \tan\beta)/\tan\alpha\tan\beta$. Change the subject to h.

21. The esplanade in a certain seaside town on the south coast lies precisely east and west and is 1500 yards long. A yachtsman wishing to know his distance from shore observed (with his compass) that the lamp at one end of the esplanade bore 11° east of north and the lamp at the other end 6° west of north. How far was the yacht out at sea?

22. On another occasion the bearings were, as before, 11° and 6°, but both were to the east of north. Where was the yacht?

23. Upon the top of a hill there is a flagstaff 42 feet high. From where I stand the angle of elevation of the bottom of the staff is 10°, that of the top 12°. How far away is the flagstaff? What is the difference of level between my eye and the top of the hill?

24. From the top of a church tower I look due north towards a river flowing east and west. The angle of depression of the further bank is 34°, that of the nearer bank 37°. The river is at this point 20 feet wide. How high is the tower (to the level of my eye)? What is the distance of the river from the tower?

25. A boat is sailing towards a cliff. At a certain point the angle of elevation of the cliff is found to be 8°. When the boat has come 200 yards nearer the angle is 13°. Find the height of the cliff.

26. A church spire is due north of a point A. From a point B 300 feet west of A the spire bears 14° east of north. What is the distance from A to the point on the ground below the top of the spire? The elevation of the top of the spire as seen from A is 9°. How high is the spire?

27. Make a paper or cardboard model to illustrate problems like No. 26. Let PN be the height of the spire. Draw the two right-angled triangles PNA, BAN. Fold the figure about AN so that the triangle PNA is vertical.

Let PN $= h$, AB $= d$, ANB $= \alpha$, PAN $= \beta$. Write formulæ for calculating (i) NA, (ii) h.

28. An iceberg was seen due east of a ship and had an elevation of 27°. After the ship had sailed a quarter of a

mile north the bearing of the berg was 47° east of south. What was its height?

TABLE OF SINES AND COSINES.

To be read thus: Sin 7° = 0·122, cos 53° = 0·602, etc.

Sin	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°
	0·00	0·017	0·035	0·052	0·070	0·087	0·105	0·122	0·139	0·156
Cos	90°	89°	88°	87°	86°	85°	84°	83°	82°	81°
Sin	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°
	0·174	0·191	0·208	0·225	0·242	0·259	0·276	0·292	0·309	0·326
Cos	80°	79°	78°	77°	76°	75°	74°	73°	72°	71°
Sin	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°
	0·342	0·358	0·375	0·391	0·407	0·423	0·438	0·454	0·469	0·485
Cos	70°	69°	68°	67°	66°	65°	64°	63°	62°	61°
Sin	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°
	0·500	0·515	0·530	0·545	0·559	0·574	0·588	0·602	0·616	0·629
Cos	60°	59°	58°	57°	56°	55°	54°	53°	52°	51°
Sin	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°
	0·643	0·656	0·669	0·682	0·695	0·707	0·719	0·731	0·743	0·755
Cos	50°	49°	48°	47°	46°	45°	44°	43°	42°	41°
Sin	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°
	0·766	0·777	0·788	0·799	0·809	0·819	0·829	0·839	0·848	0·857
Cos	40°	39°	38°	37°	36°	35°	34°	33°	32°	31°
Sin	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°
	0·866	0·875	0·883	0·891	0·899	0·906	0·914	0·921	0·927	0·934
Cos	30°	29°	28°	27°	26°	25°	24°	23°	22°	21°
Sin	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°
	0·940	0·946	0·951	0·956	0·961	0·966	0·970	0·974	0·978	0·982
Cos	20°	19°	18°	17°	16°	15°	14°	13°	12°	11°
Sin	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°
	0·985	0·988	0·990	0·993	0·995	0·996	0·998	0·999	0·999	1·00
Cos	10°	9°	8°	7°	6°	5°	4°	3°	2°	1°

EXERCISE XIX.

THE USE OF THE SINE- AND COSINE-TABLES.

Note.—Distances should be calculated to the nearest tenth of the unit, angles to the nearest half-degree. Neat diagrams should accompany solutions, but need not be drawn to scale.

A.

1. Calculate the amount of northing or southing, and of easting or westing made by a ship on each of the following occasions :—

	Course.	*Distance run.*
(i)	23° E. of N.	15 miles.
(ii)	42° W. of N.	17 miles.
(iii)	68° W. of S.	22 miles.
(iv)	7° E. of S.	35 miles.

2. After sailing 26 miles on a course E. of N. a ship is 8·1 miles farther north than at starting. Find (i) the course and (ii) the amount of easting.

3. Find the course of a ship which after sailing 18 miles between west and south, has made 14·6 miles towards the west. Calculate the southing.

4. A ship sailing 34° W. of N. has reached a point 12 miles farther north than her point of departure. Find the distance she has run and the amount of westing.

5. A ship which has been sailing 28° W. of S. has made 9·4 miles to the west. Find the distance run and the amount of southing.

6. A smack after leaving Yarmouth harbour has sailed in succession on the following courses. Find how far she is north and east of the harbour mouth :—

Courses.	*Distances run.*
82° E. of N.	5 miles.
27° E. of N.	14 miles.
72° E. of S.	18 miles.

7. Use the tangent table to find the final bearing of the ship in No. 6 from the harbour mouth. Calculate also the distance.

8. A ship took her departure from a point in the Channel where the Lizard bore 23° W. of N. and was 15 miles away. She sailed, 45° E. of S., 34 miles, and then, 79° W. of S., 16 miles. Calculate the final bearing and distance of the Lizard.

9. A ship was steered for 3 hours 40° W. of N. in a current running 60° W. of S. at 2 miles an hour. According to the log the distance run was 17 miles. Find the actual course and the actual distance run.

10. A Channel swimmer estimates that he has swum S.E. 6 miles from his starting-point at Deal, but that the tide has carried him 40° W. of S. 14 miles. How far is he now from Deal?

Note.—In the figures illustrating Nos. 1-5 let A represent the ship's starting-point, B the point at the other end of the distance run, AC the northing or southing, CB the easting or westing. It is convenient to represent the length of CB (i.e. the side opposite A) by a, the length of AC (opposite to B) by b, the length of AB (opposite to C) by c, and the course (i.e. the number of degrees in the angle CAB) by α.

11. Write in symbols the sailor's rules for finding:—

(i) The northing or southing, given the course and the distance run;
(ii) The easting or westing, given the same;
(iii) The distance run, given the northing or southing, and the course;
(iv) The distance run, given the easting or westing, and the course.

12. Give the rules for finding:—

(i) The easting or westing, given the northing or southing, and the course;
(ii) The northing or southing, given the easting or westing, and the course.

13. Give the rules for finding:—

(i) The course, given the northing or southing, and the distance run;
(ii) The course, given the easting or westing, and the distance run;
(iii) The course, given the northing or southing and the easting or westing.

B.

14. A boy scout is walking along a straight road towards the west. He leaves the road at A by a straight footpath on the left, in order to examine a tree at C, 32 yards along the path. He then returns to the road by a second straight path at right angles to the former, striking it at B, 73 yards from the place where he quitted it. Find the angles between each footpath and the road, and the length of the second path.

15. Between two points A and B in the footpath across a field there is a piece of swampy ground. To avoid it a lady leaves the path at A and walks 42 yards along a straight line to a point C. Here she turns again towards the path and rejoins it at B, 54 yards in a straight line from C. Given that AC and CB make respectively angles of 47° and 35° with AB, calculate the direct distance from A to B. [Draw a perpendicular from C to AB.]

16. In a case similar to that of No. 15, it is known that AC = 52 yards, the angle A = 39°, the angle B = 51°, and the direct distance AB = 67 yards. Calculate the distance along CB.

17. A man is walking in a north-easterly direction across a common. At a point A he turns 22° towards the left and proceeds 150 yards to read a sign-post at C. Here he turns towards the right and walks 180 yards farther to a rock B where he sits down. From his present position the angle between A and C is 24°, and A is 280 yards away. Calculate (i) the angle between AB and the north-easterly line upon which he was originally walking; (ii) the number of degrees through which he turned at C.

Note.—In a diagram intended to illustrate completely any one of the last four problems it would be necessary :—

(*a*) to draw the lines to scale to represent the given distances;

(*b*) to draw them in the proper directions, *i.e.* making the given angles with one another;

(*c*) to mark them with arrow-heads to show which way along them the movements were supposed to take place.

The movements actually taken along AC and CB may be called **component** movements: the direct movement AB, which would lead from the same starting-point to the same

final point, may be called the **resultant** movement. Straight lines drawn to represent such movements completely are called **vectors.** AC and CB are **component vectors,** AB is a **resultant vector**. Remember that the resultant vector like the component vectors should always carry an arrow-head. Care should be taken in naming a vector by letters at its ends to give the letters in the order which represents correctly the "sense" of the movement, *i.e.* the way which it takes along the line it follows. Thus the second component vector in Nos. 14-16 is called CB, not BC; the resultant vector is called AB not BA. The operation of finding the resultant of two or more component vectors is called **compounding** the vectors. When a single vector is replaced by two or more vectors of which it would be the resultant it is said to be **resolved into components.**

18. There are two vectors at right angles, a and b. The length of a is 5·6 cms., and it makes an angle of 37° with the resultant vector c. Calculate the lengths of b and c.

19. Resolve a vector, c, of length 14·8 inches into two component vectors, a and b, at right angles, so placed that a makes an angle of 52° with c.

Note.—The difference of direction between two vectors, AC, CB, is the angle through which you would turn at the moment of passing from AC to CB. It is not the angle ACB but its "supplement".

20. The difference of direction between two vectors is 69°, their lengths are 10·6 inches and 23·5 inches respectively, and the latter makes an angle of 20° with the resultant. Calculate the length of the resultant.

21. Find the resultant of two vectors of lengths 17·2 cms. and 14·6 cms. respectively, given that the former makes an angle of 48° with the resultant, and that the difference of direction between them is 109°.

22. In No. 20 resolve the vector CB = 23·5 into a vector CP along the line of AC and a vector PB at right angles to it. That is, let a point travel from A to B along the lines AP, PB, at right angles to one another, instead of along the original AC, CB. Calculate the length of PB and of CP and hence of AP. From AP and PB calculate the angle A and hence the angle B. Calculate the length of AB. Compare the results with those obtained in No. 20. What information given in No. 20 was, strictly speaking, superfluous?

23. Show that No. 21 could have been solved in a similar way if the angle of 48° had not been mentioned.

24. Two vectors of length 8·6 cms. and 13·2 cms. differ in direction by 47°. Calculate the angles which they make with the resultant, and the length of the latter.

25. Calculate the resultant of two vectors of length 8·6 inches and 4·3 inches respectively when their directions differ by 142°.

C.

26. By means of his range-finder an artillery officer discovers that one of the enemy's guns is 3200 yards and the other 2700 yards away. The angle between them is 114°. Calculate their distance from one another.

27. Two hostile warships are firing at one another. An observer on shore judges by the interval between the flash and the sound of a gun, that at a certain moment one ship is 11·3 miles and the other 8·4 miles distant from him. He judges also that the angle between them is about 20°. About how far are the ships apart?

28. Two straight high-roads, AC, CB, meet at C. A lane, also straight, cuts across from A to B, making an angle of 53° with AC and of 49° with CB. AB is 800 yards long. How much distance does a cyclist save by taking the short cut from A to B?

[Draw the vector diagram. Replace AB, as before by a vector AP along the line of AC and PB at right angles to it. The length of PB can be calculated in two ways—from the triangle APB and the triangle BPC. Hence show that $a \sin 78° = 800 \sin 53°$. Calculate a from this relation. To find b replace AB by AQ, QB at right angles, Q being on BC.]

29. Solve a problem similar to No. 28, substituting 63° and 42° for the angles and 1120 yards for the length of the lane.

30. Two railway stations, A and B, are 15 miles apart in a straight line, but the line deviates from its direction at A in order that trains may call at a third place C. Supposing that the lines AC and CB are straight and that they make respectively angles of 23° and 34° with AB, find how much longer the journey is from A to B than it would be if the line did not deviate to C.

31. Two lighthouses are exactly 8 miles from one another on a north and south line. The master of a ship who wishes

to fix his position observes (with the compass) that the northern light bears due west and the southern light 50° west of south. How far is the ship from the two lighthouses.

32. An hour later the northern lighthouse bears 80° W. of N. and the southern lighthouse 64° W. of S. How far is the ship now from the two lighthouses. (Indicate the two positions of the ship in the same diagram.)

33. A certain seaside town has a straight esplanade a mile long. From the northern end a flagstaff on an island in the sea bears 43° W. of N. From the southern end the flagstaff bears 32° W. of N. From the southern to the northern end the esplanade itself points 12° E. of N. Find the distance of the flagstaff from the northern end of the esplanade.

34. I want to find the distance from my house to a distant church whose spire is visible from my window. I measure the angle between the church and the last telegraph post I can see on the road which runs before my house, and find it to be 64°. Standing by the post I find the angle between the church spire and my window to be 51°. The distance along the road between the post and my window is, by my cyclometer, 0·6 mile. How far is the church from my house?

EXERCISE XX.

SOME NAVIGATION PROBLEMS.

A.

Note.—The word "miles" in this Exercise always means nautical miles. To find sin 24° 20′ *add* to sin 24° one-third of (sin 25° − sin 24°). To find cos 24° 20′ *subtract* from cos 24° one-third of (cos 24° − cos 25°).

1. A steamer left the Island of Ascension (lat. 7° 56′ S.) for Tristan d'Acunha and sailed 1600 miles due south. Find its latitude.

2. Another vessel sailed due north from Ascension with the object of making the African coast near Sierra Leone. Find its latitude after a run of 1200 miles.

3. A ship sailing from Halifax (Nova Scotia—lat. 44° 36′ N.) to the Bermudas (32° 20′ N.) follows practically a southern course. Find the distance.

4. Auckland (lat. 50° 40′ S.) and Bering Island (lat. 55° N.) are on the same meridian. How far are they apart?

5. When Captain Scott's Antarctic expedition left Christchurch, New Zealand (lat. 43° 30′ S.) in 1911, how far was it from the South Pole?

6. An exploring expedition circumnavigated the Antarctic seas along the sixtieth parallel. What was the distance travelled?

7. New York (long. 74° W.) and Oporto (8° 31′ W.) are both very nearly on the forty-first parallel (N.). What is their distance apart along this parallel?

8. A ship sailing from Sydney (long. 151° 12′ E.) to Valparaiso (71° 30′ W.) could follow the thirty-third parallel (S.) for practically all the way. Calculate the length of the voyage.

Note.—A sailor can determine his latitude wherever he is by simple observations of the sun or the stars. To find his longitude he needs, as a rule, a chronometer which will tell him at any moment what the time is at Greenwich. Before

Harrison's invention of the chronometer (1736) determinations of longitude were troublesome. Consequently the sailor seeking a distant port preferred, if possible, to sail due north or south until he reached the latitude of the port and then to sail east or west along the parallel until he came to it. This method of navigation is called **parallel sailing.**

9. A ship leaves Boston (42° 25′ N., 71° W.) for Barbados (13° N., 59° 45′ W.) and uses parallel sailing. Calculate the southing and the easting.

10. Find the southing and westing of a ship which goes by parallel sailing from Bombay (18° 55′ N., 72° 54′ E.) to Cape Town (33° 40′ S., 18° 30′ E.).

B.

Note.—In Nos. 11-13 no allowance need be made for *change* of latitude.

11. A ship in lat. 54° N., long. 34° 16′ W., sails 24 miles 38° W. of N. Find its new latitude and longitude.

12. A ship sails from lat. 63° S., long. 43° W. to lat. 63° 40′ S., long. 42° 18′ W. Find the southing, easting, course and distance run.

13. A ship sails from lat. 46° N., long. 164° 18′ E., and reaches the 47° parallel in long. 163° 30′ E. Calculate the course and the distance run.

Note.—Nos. 14-18 are to be taken as examples of **middle latitude sailing.**

14. Find the course and the distance between Cape Clear (lat. 51° 25′ N., long. 9° 29′ W.) and Brest (lat. 48° 23′ N., long. 4° 29′ W.).

15. Find the latitude and longitude of a ship after it has sailed from Brest, 67° W. of S., 200 miles.

16. From a place in lat. 48° N., long. 25° W., a ship sails (roughly) south-easterly 215 miles until her **departure** from the meridian (i.e. her easting) is 167 miles. Find the course steered and the new latitude and longitude.

17. Another ship sails from the same place and in three days reaches lat. 52° N., having made a departure from the meridian of 260 miles to the west. Calculate her course, distance run, and longitude.

18. A ship sails north-easterly from lat. 50° S., long. 80° E., 330 miles, and finds herself in lat. 46° S. Find the course steered and the new longitude.

EXERCISE XXI.

RELATIONS OF SINE, COSINE, AND TANGENT.

A.

1. Use the relation $\tan \alpha = \sin \alpha/\cos \alpha$ to test the concordance of the ratios given in the tables on pp. 107, 111 for the angles 12°, 36° and 64°.

2. Test by means of the relation $\sin^2 \alpha + \cos^2 \alpha = 1$ the concordance of the values given in the table on p. 111 for 20°, 30°, 53°.

3. The sine of an angle is $\frac{5}{13}$. What is its cosine?

4. I am told that the sine and cosine of a certain angle are respectively 0·18 and 0·88. Is the information correct?

5. Use the relation $\tan \alpha = \sin \alpha/\sqrt{(1 - \sin^2 \alpha)}$ to test the agreement of the values given for sin 35° and tan 35°.

6. The sine of an angle is $\frac{12}{37}$. Calculate its tangent as a vulgar fraction.

7. Use the relation $\tan \alpha = \sqrt{(1 - \cos^2 \alpha)}/\cos \alpha$ to test the values given for cos 56° and tan 56°.

8. The cosine of an angle is $\frac{7}{25}$. Calculate its tangent.

9. Apply the test $\cos \alpha = 11\sqrt{(\tan^2 \alpha + 1)}$ to the values given for tan 32° and cos 32°.

10. The tangent of an angle is $\frac{13}{84}$. Calculate its cosine.

11. Apply the test $\sin \alpha = \tan \alpha/\sqrt{(\tan^2 \alpha + 1)}$ to the values given for tan 66° and sin 66°.

12. The tangent of an angle is $\frac{16}{63}$. Calculate the sine.

B.

13. Prove by a figure that (i) $\tan \alpha = \sin \alpha/\cos \alpha$,
(ii) $\sin \alpha = \cos \alpha \tan \alpha$, (iii) $\cos \alpha = \sin \alpha/\tan \alpha$.

14. Prove that $\tan \alpha/(1 - \tan \alpha) = \sin \alpha/(\cos \alpha - \sin \alpha)$. Verify by substituting the values given in the tables for 28°. Why does this formula apply only to angles less than 45°?

Does it hold good for 45° itself? How must the formula be written for angles greater than 45°?

15. Prove that $\dfrac{1 - \tan\alpha}{1 + \tan\alpha} = \dfrac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}$ for angles less than 45°. Verify when $\alpha = 24°$. Rewrite the formula to suit angles greater than 45°.

16. Prove in two ways that $\cos^2\alpha + \sin^2\alpha = 1$. Deduce formulæ for calculating (i) $\cos\alpha$ when $\sin\alpha$ is known; (ii) $\sin\alpha$ when $\cos\alpha$ is known.

17. Prove that $\cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$.

18. Prove that $(\cos\alpha - \sin\alpha)(\cos\alpha + \sin\alpha) = 1 - 2\sin^2\alpha$. Verify by substitution when $\alpha = 0°, 15°$. Modify the formula to suit angles greater than 45°.

19. Prove that $\dfrac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \cos^2\alpha - \sin^2\alpha$.

20. Prove that $\dfrac{1 - \tan\alpha}{1 + \tan\alpha} = \cos^2\alpha - \sin^2\alpha$. Verify when $\alpha = 38°$ and (after modifying the formula) when $\alpha = 63°$.

21. Demonstrate the following equivalences:—

(i) $\tan\alpha = \sin\alpha/\sqrt{(1 - \sin^2\alpha)}$.

(ii) $\tan\alpha = \sqrt{(1 - \cos^2\alpha)}/\cos\alpha$.

(iii) $\cos\alpha = 1/\sqrt{(1 + \tan^2\alpha)}$.

(iv) $\sin\alpha = \tan\alpha/\sqrt{(1 + \tan^2\alpha)}$.

22. Calculate to three places of decimals the values of the sine, cosine, and tangent of 30°, 45°, 60°.

EXERCISE XXII.

LINEAR RELATIONS.

Note.—The expression (5, 13) means that the value of x is 5 and the corresponding value of y is 13.

A.

1. The following pairs of values of x and y are connected by linear relations. Apply a test to each pair to find the form of the relation.

(i) (5, 13), (12, 27). (ii) (3·5, 10·5), (6·7, 20·1).
(iii) (8, 27), (15, 6). (iv) (4, 2·1), (10, 8·7).

2. Find the full form of the relation in each of the foregoing examples. Draw (on one sheet of paper) the graphs in (iii) and (iv).

3. Find the linear relations connecting the following pairs of values of x and y, using the **composition** method and the **substitution** method alternately.

(i) (3, 14), (7, 7·2). (ii) (5·4, 3·0), (9·6, 15·6).
(iii) (0·7, 1·6), (8·7, 12). (iv) (5, 2·3), (12, 20·5).

4. Two variables are connected by a relation of the form $y = a + b/x^2$. When $x = 2$, $y = 3$; when $x = 4$, $y = 2·1$. Find the relation. Calculate the value of y when $x = 1$ and when $x = 10$.

5. Two variables are connected by a relation of the form $y = a\sqrt{x} - b$. When $x = 4, y = 8·5$; when $x = 25, y = 25·3$ Find the relation and the value of y when $x = 64$.

6. The relation between two variables has the form :—

$$y = 1 + \frac{a}{1+x} + \frac{b}{1+x^2}.$$

When $x = 1$, $y = 5$; when $x = 2, y = 3$. Find the relation. What is the value of y when $x = 3$?

7. Explain the tests which you apply to determine whether a linear relation is of the form $y = bx$, $y = a - bx, y = a + bx$ or $y = bx - a$.

8. A linear relation is of the form $y = a + bx$. Describe

in symbols (i) the composition method, (ii) the substitution method of calculating b in a given case. Give also a formula for a. Use (P, Q) and (p, q) as symbols for the given pairs of values of x and y.

B.

9. Find by the substitution method the simultaneous values of x and y which are common to the following pairs of linear relations:—

(i) $y = 9 - 3x$, $y = 4x - 5$.
(ii) $y = 1{\cdot}8 + 5x$, $y = 11{\cdot}4 - 3x$.
(iii) $y = 2x - 3{\cdot}4$, $y = 2{\cdot}3x - 6{\cdot}4$.
(iv) $y = 24{\cdot}8 - 2x$, $3x - 2y = 5$.
(v) $y = \frac{1}{3}x + 7$, $\frac{1}{4}x - \frac{1}{3}y = 1$.

Confirm your answers to (i), (ii), and (iv) by graphs.

10. A cyclist leaves his home which is 20 miles from Charing Cross at 9 a.m. and rides towards London at 10 mls./hr. A motorist starts for London at the same moment from a place 30 miles from Charing Cross, and travels 18 mls./hr. At what time does the motorist overtake the cyclist and where?

11. The water in a certain reservoir is 6 feet deep, but the level is sinking 4 inches per day. The water in another reservoir is 3 feet deep and is rising 5 inches per day. When will the depth in the two reservoirs be the same, and what will that depth be?

12. Two vertical spiral springs hang side by side. When weights of 5 grms. are hung at their ends the first is 26 cms. and the second 20 cms. long. When the weight of 5 grms. is replaced by one of 10 grms. their lengths are 30 cms. and 28 cms. respectively. What weight will cause them to have the same length and what will that length be?

13. Calculate by the composition method the simultaneous values of x and y which are common to the following pairs of linear relations:—

(i) $4x + 3y = 25$, $5x - 3y = 11$.
(ii) $2x - 3y = 5$, $3y - x = 8$.
(iii) $5x - 2y = 11$, $x + 4y = 33$.
(iv) $7x + y = 20$, $2x + 3y = 20{\cdot}1$.
(v) $3x + 4y = 67$, $4x - 3y = 6$.
(vi) $7x + 11y = 36$, $13x + 10y = 46$.
(vii) $2{\cdot}3x - 0{\cdot}7y = 3{\cdot}4$, $0{\cdot}3y - 0{\cdot}4x = 0{\cdot}3$.
(viii) $4{\cdot}6x - 1{\cdot}3y = 2{\cdot}2$, $6{\cdot}9x - 2{\cdot}7y = 1{\cdot}8$.

Your results should be tested in each case by substitution.

14. Illustrate by graphs your answers to No. 13, (i) and (vii).

15. Solve No. 13, (ii), (iv), and (vii) by the substitution method.

16. Two delicate spiral springs hang side by side. When they are loaded with weights of 5 grms. their lengths are 31·3 cms. and 40·8 cms. respectively. When weights of 10 grms. are substituted the lengths become 46·8 cms. and 64·3 cms. Find (i) by drawing a graph and (ii) by a calculation whether any weight will stretch them equally—assuming a linear relation between length and load.

17. A man said: "I am thinking of two numbers; four times the first added to five times the second gives 7; twice the first subtracted from six times the second leaves 22". Is it possible to find two such numbers?

18. Find the values of x and y which satisfy simultaneously the following pairs of relations:—

(i) $\frac{1}{2}x^2 + \frac{3}{8}y^2 = 6,\ 5x^2 - 6y^2 = 21.$

(ii) $\frac{3}{x} - \frac{4}{y} = 0,\ \frac{3}{x} + \frac{4}{y} = 4.$

(iii) $\frac{6}{x} - \frac{7}{y} = 2,\ \frac{2}{x} + \frac{14}{y} = 3.$

(iv) $\frac{12}{x^2} - \frac{1}{3y^2} = 1,\ \frac{6}{x^2} + \frac{1}{3y^2} = 1.$

(v)[1] $x^2 - y^2 = 16,\ x + y = 8.$

(vi) $4x^2 - 9y^2 = 63,\ 2x - 3y = 3.$

(vii) $x^2 + 2xy + y^2 = 49,\ 2x - 3y = 4.$

(viii) $4x^2 - 12xy + 9y^2 = 4,\ 4x - 7y = 3.$

C.

19. Eliminate the variable z from each of the following pairs of relations:—

(i) $3x - 4y + 2z = 1,\ x + y - z = 0.$

(ii) $7x + 3z = 2,\ 2z + 5y = 1.$

(iii) $\frac{1}{3}z - \frac{1}{4}x - y = \frac{1}{8},\ \frac{3}{4}x = \frac{2}{3}z.$

(iv) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2},\ \frac{1}{3x} + \frac{1}{2y} - \frac{1}{z} = 0.$

[1] If $x^2 - y^2 = 16$ and $x + y = 8$ what is the value of $x - y$? When you know the value of $x + y$ and $x - y$ you can find the value of x and y.

20. From your answer to No. 19 (i) find (i) the value of y when $x = 2$; (ii) the value of x when $y = 12$. Find the corresponding values of z. Do both relations give the same value for z in each case?

21. In No. 19 (iii) what value of x is associated with $y = 1\frac{1}{2}$? Verify that these values of x and y are associated with the same value of z.

22. Eliminate a from the three relations:—

$$x - y = \sin a, \; x + y = \cos a, \; y/x = \tan a.$$

23. From the following pairs of relations derive statements involving x and y only:—

(i) $x = 3 \sin a$, $y = 4 \cos a$.
(ii) $a = x \sin a$, $b = y \cos a$.
(iii) $\sqrt{(x + y)} \sin a = 1$, $\sqrt{(x - y)} \cos a = 1$.

24. In No. 23 (i) find the value of y associated with $x = 2\sqrt{2}$. What value of a is associated with these values of x and y?

25. In No. 23 (iii) what is the value of y when $x = 4$?

26. Eliminate a from the relations:—

$$\sqrt{(2x - 3y)} = \tan a, \; \sqrt{(3x - 2y)} \cos a = 1$$

27. In the last example find the value of x when $y = \frac{1}{3}$. Find the corresponding value of a.

EXERCISE XXIII.

INVERSE PROPORTION.

A.

1. A man and his son both determine to save money regularly—the former to buy a motor-bicycle costing £40, the latter to buy an ordinary bicycle costing £10. Draw on the same sheet two graphs exhibiting the average savings, in shillings per week, necessary to provide the required sums in a given number of weeks. (Choose your scales so as to include a saving by the boy of sixpence a week.)

2. Let the formulæ corresponding to the graphs in No. 1 be $S = k_1/t$ and $S = k_2/t$. What will be the value of k_1 and k_2 if S represents the savings (i) in shillings per week, (ii) in pounds per month of four weeks, (iii) in pence per day?

3. Two variables, x and y, are in inverse proportion. When $x = 2{\cdot}7$ $y = 10{\cdot}8$. Write down the relation between them. What is the value of y when $x = 0{\cdot}018$ and the value of x when $y = 0{\cdot}0002$?

4. If the graph of the relation of No. 3 were drawn what would be (i) the co-ordinates of the vertex, (ii) the length of the axis?

5. What pair of values of x and y satisfies simultaneously (i) the relations $y = 2x$ and $xy = 200$; (ii) the relations $y = \frac{1}{2}x$ and $xy = 800$?

6. Illustrate your answer to No. 5 by drawing across the graphs of No. 1 the straight lines corresponding to $y = 2x$ and $y = \frac{1}{2}x$.

7. Show that the pair of values $(p/q, pq)$ is common to the relations $xy = p^2$ and $y = q^2x$. At what point would the corresponding graphs cross one another?

8. Prove by algebra that two inverse proportion curves can never intersect. [Take $xy = a$ and $xy = b$ as the corresponding formulæ and show that they allow no common values of x and y.]

9. Regard the curves in No. 1 as corresponding to the relations $xy = a$ and $xy = b$. At a certain point on the x-axis let the ordinates of the two curves be P and Q. At a certain point on the y-axis let the abscissæ be p and q. Show that wherever these points are taken, $P/Q = a/b = p/q$.

10. What is the ratio of the axes of these curves?

B.

11. A number of cylindrical tins of different shapes are to be made, each to hold a pint of liquid (34·7 cubic inches) when filled to 1 inch from the top. Describe in words the relation between the height and the area of the bottom of the tins. Express the relation in a formula (h, A). Calculate the height of the tin in which the bottom contains 6·94 square inches.

12. In the case of another set of tins I know that each is to carry 100 cubic inches but I do not know how much free space is to be left above the liquid. Taking up one of the tins I find that its height is 10 inches and the area of cross-section 12 square inches. Up to what point is the vessel intended to be filled? The sectional area of another of the set is 20 square inches. What is its depth?

13. Write formulæ descriptive of the relation between x and y:—

(i) When $(y - 3)$ is inversely proportional to x;
(ii) When y is inversely proportional to $(x + 4{\cdot}7)$;
(iii) When $(y + 2)$ is inversely proportional to $(x - 5)$.

Use k in each case as the symbol for the unknown constant.

14. Rewrite the formulæ of No. 13, replacing k by its numerical value calculated from the following information:—

In (i) when $x = 4$, $y = 10$.
In (ii) when $x = 5{\cdot}3$, $y = 0{\cdot}36$.
In (iii) when $x = 13$, $y = 7$.

15. A variable y is inversely proportional to $(x - a)$. When $x = 12$, $y = 10$ and when $x = 10$, $y = 15$. Find the formula for the relation.

16. The following relations between x and y may be regarded as expressing inverse proportion between certain numbers. Describe the numbers.

(i) $xy + 5x + 2y = 20$. (ii) $xy - 3x + 7y = 121$.
(iii) $6xy + 4x + 9y = 132$. (iv) $5xy - 100x - 2y = 0$.

17. In No. 16 (i) find the value of y when $x = 3$. In (iii) find x when $y = 10$. In (iv) find x when $y = 30$.

18. What pair of values of the variables is common to the relations $y = x + 2$ and $xy = 8$? [After substitution use the method of Ex. XVI, No. 118.]

19. Find the values of x and y which satisfy simultaneously the relations $y = 4x - 4$ and $xy = 8$.

20. Illustrate your answers to Nos. 18 and 19 by drawing on one sheet the graphs corresponding to $y = x + 2$, $y = 4x - 4$, and $xy = 8$.

EXERCISE XXIV.

PROPORTION TO SQUARES AND CUBES.

A.

1. Obtain a graph showing the breaking load of a hempen rope (Ex. IV, No. 19) by first drawing the straight line $y = 0{\cdot}6x$ and then transforming it into the curve[1] $y = 0{\cdot}6x^2$. (Graduate the x-axis from 0 to 30.) From the graph determine (i) the breaking load when the circumference of the rope is 2·5 inches, (ii) the circumference of the rope that will just support 7·8 tons.

2. Obtain a graph showing the distances visible from a given height (Ex. IV, No. 6) by transforming the straight line $y = 1{\cdot}22x$ into the curve $y = 1{\cdot}22\sqrt{x}$. (Graduate as before.) Use it to find (i) the distance visible from a height of 2000 feet, (ii) the height at which you can see 50 miles.

3. Describe in words the relation between p and L in Ex. IV, No. 5. Obtain a graph of the relation by transforming the line $y = 0{\cdot}1x$ into the curve $y = 0{\cdot}1x^3$. From the graph find (i) p when $L = 2{\cdot}8$, (ii) L when $p = 2$.

4. The diameter in centimetres of a certain kind of spherical bullets is given by the formula $d = 0{\cdot}8\sqrt[3]{w}$, w being the weight of the bullet in grammes. Obtain a "ready-reckoner" of the diameters of these bullets by transforming the line $y = 0{\cdot}8x$ into the curve $y = 0{\cdot}8\sqrt[3]{x}$. Find (i) the diameter of a bullet weighing 35 grms., and (ii) the weight of the bullet with a diameter of 1·7 cm.

5. Write formulæ descriptive of the following relations:—

(i) y is directly proportional to the square of x and when $x = 7$, $y = 4{\cdot}9$;

(ii) y is directly proportional to the square root of $x + 4$ and when $x = 0$, $y = 2{\cdot}8$;

(iii) $y - 3$ is directly proportional to the cube root of $x + 7$ and when $x = 20$, $y = 18$.

[1] "The curve $y = 0{\cdot}6x^2$" is, of course, a shortened expression for "the curve corresponding to the relation $y = 0{\cdot}6x^2$".

Note.—Suppose you have two statements such as :—

$$16 = a\sqrt{(30 + b)} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$8 = a\sqrt{(20 + b)} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

They can be reduced to a single statement containing b but not a by dividing the two sides of (i) by the corresponding sides of (ii) :—

$$2 = \sqrt{(30 + b)}/\sqrt{(20 + b)} \quad . \quad . \quad \text{(iii)}$$

From (iii) b can be calculated, while a can then be found from (ii) or (i).

6. It is known that in a certain relation y is directly proportional to the square root of $x - a$, that, when $x = 10$, $y = 6$ and that, when $x = 17$, $y = 8$. Find the relation.

7. A variable y is directly proportional to the square of $x + a$. When $x = 0$, $y = 36$, when $x = 1$, $y = 64$. Find the relation.

8. What simultaneous values of x and y are common

(i) to the relations $y = 0{\cdot}6x^2$ and $y = 3x$;
(ii) to the relations $y = 0{\cdot}8\sqrt[3]{x}$ and $y = 0{\cdot}2x$;
(iii) to the relations $y = 3\sqrt{x}$ and $y = 2\sqrt{(x + 10)}$;
(iv) to the relations $y = 2(x + 3)^2$ and $y = 2(x - 7)^2$?

Illustrate your answers to (i) and (ii) by means of the graphs of Nos. 1 and 4.

9. Draw a graph to illustrate your answer to No. 8 (iii).

B.

10. Describe in words the relation of P and D in Ex. IV, No. 30. Construct a "ready-reckoner" for the formula by transforming the curve $xy = 18$ into the curve $y = 18/\sqrt{x}$. Find (i) the loss of steam-pressure with a cylinder 20 inches in diameter, and (ii) the diameter of cylinder that gives a loss of 3·6 lb./in.^{2}

11. When a long straight magnet is held vertically above a piece of iron (e.g. a nail on a table) its lifting power is inversely proportional to the square of the distance between the iron and the bottom of the magnet. A certain magnet can just lift a nail weighing 12 grms. at a distance of 1 cm. Write down the relation between w and d.

12. Exhibit the relation of No. 11 in a graph obtained by transforming the curve $xy = 12$ into the curve $y = 12/x^2$.

13. Express in formulæ the following relations :—

(i) y is inversely proportional to the square of $x - 3$ and when $x = 5$, $y = 3$;
(ii) $y - 4$ is inversely proportional to the square root of $x - 10$ and when $x = 11$, $y = 6$;
(iii) y is inversely proportional to the cube of x and when $x = 2$, $y = 3$;
(iv) y is inversely proportional to the cube root of x and when $x = 64$, $y = 6$.

14. On the same sheet of paper transform the curve $xy = 24$ into the curves which correspond to the relations of No. 13 (iii) and (iv).

15. If the curves corresponding to the relations of No. 13 (i) and (ii) were drawn what lines would be, respectively, their asymptotes?

16. Find the relations that satisfy the following conditions:—

(i) y is inversely proportional to the square of $x + a$; when $x = 0$, $y = 12$ and when $x = 2$, $y = 3$;
(ii) y is inversely proportional to the square root of $2x - a$; when $x = 2$, $y = 21$ and when $x = 6$, $y = 7$.

17. Find the pairs of values of x and y which are common

(i) to the relations $y = 12/\surd x$ and $y = 18/\surd(x + 5)$;
(ii) to the relations $y = 3\surd(x - 6)$ and $y = 24/\surd(x + 6)$;
(iii) to the relations $y = 4/\surd(x^2 - 100)$ and $y = 1/\surd(x + 10)$.

18. Illustrate your answer to No. 17 (ii) by drawing the graphs of the relations.

EXERCISE XXV.

JOINT VARIATION.

1. Describe in words the relations between the variables in the following examples from Ex. IV: (i) No. 17, (ii) No. 18, (iii) No. 22, (iv) No. 25 (i), (v) No. 32.

2. Express the following relations in formulæ, putting k for the unknown constant:—

(i) z varies directly as x^2 and inversely as y;
(ii) z varies directly as the square root of x and inversely as y^2;
(iii) z is directly proportional to $\sqrt[3]{x}$ when y is constant and inversely proportional to y when x is constant;
(iv) z is inversely proportional to x when y is constant and to the cube root of y when x is constant.

3. Supply the values of k in No. 2 from the following information:—

(i) when $x = 1{\cdot}2$, $y = 0{\cdot}2$ and $z = 7{\cdot}2$;
(ii) when $x = 25$, $y = 4$ and $z = 15$;
(iii) when $x = 8$, $y = 0{\cdot}7$ and $z = 6$;
(iv) when $x = 1{\cdot}4$, $y = 1$ and $z = 0{\cdot}5$.

4. The east and west distance between two points on a globe which are situated on the same parallel of latitude is directly proportional to their difference of longitude (l), to the cosine of the latitude (λ), and to the radius (r) of the globe. Two points on the earth's surface whose latitude is 52° and whose difference of longitude is 13° are 553 miles apart. How far apart are two points on the surface of the moon whose latitude is 37° and whose difference of longitude is 24°? The radius of the earth is 3959 miles; of the moon 1080 miles.

5. To measure the strength of a bar-magnet the electrician lays it on a line east and west pointing to the middle of a small compass needle. He then notes the number of degrees (a)

through which the needle is deflected and the distance (d) of the middle of the bar-magnet from the middle of the needle. The rule is that the strength of the magnet (M) varies directly as the cube of the distance and directly as the tangent of the angle of deflection.

When a magnet of strength 6444 is placed with its mid-point 80 cms. from a needle the deflection is 4°. What is the strength of a magnet which when placed 60 cms. from the needle produces a deflection of 9°?

6. Kepler (*c.* 1610) discovered that the square of the time (T) taken by a planet to revolve about the sun varies as the cube of its distance (d) from the sun. The earth (whose revolution takes, of course, a year) is distant from the sun about 93 millions of miles. Find (i) the distance from the sun of Jupiter whose revolution takes twelve years; and (ii) the time of revolution of Neptune whose distance is about 2780 millons of miles. (Take a million miles as unit.)

7. A variable y is the sum of two parts of which the first is directly proportional to x and the second is inversely proportional to the square of x. Give a formula for y, using a and b as symbols of the two constants needed.

8. Replace a and b in No. 7 by their numerical values obtained from the knowledge that when $x = 4$, $y = 5$ and when $x = 10$, $y = 5{\cdot}08$.

9. A variable y is the difference between two other quantities. Of these the first varies directly as $\sin \alpha$ and inversely as $x - 1$; the second varies directly as $\cos \alpha$ and x conjointly and inversely as $(x - 2)^2$. Express the relation in a formula.

10. Replace the symbols of the constants in No. 9 by their numerical values, given that when $\alpha = 26°$ and $x = y$, and that when $\alpha = 33°$ and $x = 4$, $y = 2{\cdot}89$.

11. If a current of electricity is flowing round a circular coil of wire fixed vertically with its plane north and south and a small compass needle is brought near to it along the axis of the coil the needle is deflected from its usual north and south position. The tangent of the deflection (α) varies directly as the strength of the current (C), the area of the coil (A) and the number of turns of wire (n) in the coil conjointly. It also varies inversely as the cube of the distance of the needle from the plane of the coil. When a current of 5 units ("ampères") flows round a coil containing 100 turns and offering an area of 300 sq. cms. a needle at a distance of 30 cms. is deflected

through $5\frac{1}{2}°$. What deflection will be produced at a distance of 60 cms. by a current of 8 ampères flowing in a coil containing 300 turns and having an area of 200 sq. cms.?

12. Write proofs of the following propositions :—

(i) If $z \propto x$ (y constant) and $z \propto y$ (x constant) then $z \infty xy$;
(ii) If $z \propto x^2$ (y constant) and $z \propto 1/y^3$ (x constant) then $z \propto x^2/z^3$.

EXERCISE XXVI.

SUPPLEMENTARY EXAMPLES.

A. Test Paper 1.

1. The speed with which waves travel across the sea is given by the formula :—

$$s = \sqrt{32{\cdot}2d\left(1 + \frac{3h}{d}\right)}.$$

$s \equiv$ speed in ft./sec. ; $d \equiv$ depth of water in feet ; $h \equiv$ height of wave in feet.

(i) Do high waves or low waves travel more rapidly in water of the same depth ?
(ii) What formula would be sufficient to calculate s for waves moving on very deep water ?
(iii) Calculate the speed of a wave 2 feet high where the sea is 10 feet deep ;
(iv) Calculate the speed of a wave 4 feet high where the sea is 24 feet deep.

2. From the formula of No. 1 obtain a rule for calculating the depth of the sea by observing the speed of waves of a known height.

Waves 2 feet high are observed to travel across the sea at a speed of 32·2 ft./sec. What is the depth of the sea at that point ?

3. Factorize :—

(i) $ab - 3a + 2b - 6$; (ii) $p(x - 2y) + 3q(x - 2y)$;
(iii) $p^2 + 2pq + q^2 - p - q$; (iv) $9 - (2 - p)^2$.

4. Find the values of n which satisfy the following relations :—

(i) $2n = 3(n - 6{\cdot}2)$; (ii) $\dfrac{1}{2n + 3} = \dfrac{1}{5n - 9}$;
(iii) $(n - 2)(2n - 3) = 2(n - 1)^2$; (iv) $n^2 - 6n = 16$.

5. Show (i) that the square of an odd number is always odd, and (ii) that the product of any two odd numbers is always odd. [Expressions such as $2n + 1$, $2m - 1$, describe odd numbers whether n and m are themselves odd or even.]

6. When 2·3 has been subtracted from a certain number the square root of the residue is 3. What is the number?

7. The sides of an oblong table are a feet and b feet long. In damp weather the wood swells, but unevenly in the two directions. Each inch in the longer side of the table becomes $(1 + p)$ inch and each inch in the shorter side $(1 + q)$, p and q being, of course, very small. Write down an expression for the increase of area of the table to a first approximation.

8. Find the linear relations between x and y which are satisfied (i) by the values (4, 5) and (5, 7); (ii) by the values (0, 3) and (1, 1). Have the relations a common pair of values of x and y? Illustrate your answer by graphs.

9. The road from one village A to another B bears 27° E. of S. for $2\frac{1}{2}$ miles. It then turns 12° further to the east and continues in this direction for 2 miles. Find the direction from A to B as the crow flies.

10. A ship in latitude 54° N., longitude 17° W., received by wireless telegraphy a message of distress from a ship in longitude 20° W. on the same parallel. How far is the distressed ship due west of the other?

B. Test Paper 2.

1. When a submarine mine explodes near a ship it produces a tremendous pressure on its side. (To produce a fatal injury to an ironclad the pressure should be $>$ 12,000 lb./in.^{2}) The following formula shows how to calculate the pressure produced at a point A on the side of the ship, the centre of the charge being at a point B. Let $d \equiv$ distance AB, $a \equiv$ the inclination of the line AB to the horizontal, C $\equiv$ the weight of the charge in lb. K and k are constants (or coefficients) which depend on the explosive used. For No. 1 dynamite (the best explosive for the purpose) K = 100, k = 20 (also for guncotton). For gunpowder K = 25, k = 35.

$$P = \frac{9KC}{d}\left(1 + \frac{25}{d^2}\right)\left(1 + \frac{ka}{90}\right).$$

(i) What does the formula become when the points A and B are on the same level?

(ii) If AB is horizontal, what number of pounds of gunpowder is as effective as 1 lb. of dynamite?

(iii) Would this quantity of gunpowder produce a greater or less pressure than the 1 lb. of dynamite when the point A is at a higher level than B?

(iv) A charge of 100 lb. of No. 1 dynamite is exploded near a warship. Find the pressure at a point on the same level as the mine and 5 feet away. Will it be sufficient to do a vital injury to the ship?

(v) Calculate the pressure produced by the same charge at a point 10 feet away and 20° above the level of the charge.

2. The following formula is taken from an electrical engineer's pocket-book. Change the subject (i) to C_1, (ii) to C_2, (iii) to C_3. How can you determine the third formula when you know the second?

$$R_1 = R\left(\frac{2C_1}{C_2 + C_3} - 1\right).$$

3. The Egyptians used the fact that $3^2 + 4^2 = 5^2$ as a means of drawing a right angle. Show that if $a = 4n$ and $b = 4n^2 - 1$ (n being any whole number), then $a^2 + b^2$ is always the square of a whole number. Use your proof to construct a table of the first four triads which have the same property as 3, 4, and 5.

4. Transform each of the following expressions into a shape more convenient for computation:—

(i) $(2a - 3b)^2 - 16c^2$; (ii) $(3p - 2)^2 - (2q + 3)^2$;

(iii) $\dfrac{p}{qa - b} - \dfrac{q}{pa - b}$; (iv) $\dfrac{1}{a - 4} + \dfrac{2a - 1}{a^2 - 6a + 8}$.

5. Find values of n which fit the following statements:—

(i) $\frac{1}{2}(n - 2) + \frac{1}{9}(5 - n) + \frac{5}{6}(n - 1) = 0$;

(ii) $\dfrac{1}{n} - \dfrac{1}{n + 2} = \dfrac{1}{24}$;

(iii) $\dfrac{2}{5\sqrt{(9 - 2n)}} = \dfrac{1}{\sqrt{8n + 5}}$;

(iv) $\dfrac{1}{n + a} + \dfrac{1}{n - a} = \dfrac{1}{n^2 - a^2}$.

6. On a windy day a boy cycled to a place 24 miles away and returned by the same road. He estimated that the wind added 2 miles an hour to his speed on the outward journey and retarded him by the same amount on the way home. He took an hour longer to return than to go. At what rate does he normally ride?

7. The relations $y = 3x - 5$ and $y = 1 + 9/x$ include a common pair of values of x and y. Find it. Illustrate your

result by drawing the graphs of the relations (from $x = 0$ to $x = 5$) on one sheet.

8. Show that

$$\frac{(p - q)x}{1 - (p + q)x + pqx^2} = \frac{1}{1 - px} - \frac{1}{1 - qx}.$$

Use this equivalence to find a formula for the value of

$$\frac{(p - q)x}{1 - (p + q)x + pqx^2}$$

to the second approximation when x is a small fraction.

9. The relation between two variables is known to be of the form $y = a - b \sin x°$. When $x = 14\frac{1}{2}$, $y = 8$; when $x = 30$, $y = 6$. Find the relation and calculate (i) the greatest value of y; (ii) the least value of y; (iii) the value of y when $x = 56$.

Illustrate your answer by *sketching* the graph of the relation.

10. A church spire is in the form of a pyramid 24 feet high, standing on a square base which measures 14 feet each way. Calculate to the nearest degree the angles (i) between two opposite faces; (ii) between the two sloping edges of one of the triangular faces; (iii) between two opposite sloping edges of the pyramid.

[Either construct a paper model or draw sections of the pyramid to help you in your reasoning.]

C. Test Paper 3.

1. The speed with which water is carried along a pipe depends partly on the size and length of the pipe and partly on the "head" of the water. By the head is meant the vertical height above the pipe of the water surface in the cistern or reservoir. The number of gallons discharged per minute is given by the formula :—

$$G = 29{\cdot}4\sqrt{\frac{Hd^5}{l}}.$$

$H \equiv$ head in feet; $d \equiv$ diameter of pipe in inches; $l \equiv$ length of pipe in feet.

(i) Calculate the rate of discharge from a 2-inch pipe 3200 feet from the reservoir under a head of 100 feet;

(ii) The same, with an 8-inch main 5 miles long and a head of 132 feet.

2. Change the subject of the formula of No. 1 (i) to l, (ii) to H, (iii) to d.

[You are given that $1/864{\cdot}4 = 0{\cdot}001157$ and that the fifth root of this number is $0{\cdot}2586$.]

3. If $a = 2n + 1$ and $b = 2n(n + 1)$, n being any whole number, show that $a^2 + b^2$ is always the square of the number $b + 1$.

What triads are obtained by putting $n = 1, 3, 10$, in succession?

4. Factorize $(n^2 + 3n - 2)^2 - (n^2 - 3n + 2)^2$, and use the result to find two distinct values of n which reduce the value of the original expression to zero. Confirm your conclusion by substitution in the original expression.

5. Find values of n which comply with the following conditions:—

(i) $\frac{1}{6}(7n - 1) + \frac{1}{4}(3n - 1) - \frac{1}{3}(4n - 1) = 0$;

(ii) $p(n - p) - q(n - 2p) = q^2$;

(iii) $\dfrac{n + 2}{n - 4} - \dfrac{n + 1}{n - 3} = 1$.

6. One customer buys 14 lb. of tea and 10 lb. of coffee for £2 3s., and another buys 11 lb. of tea and 15 lb. of coffee for £2 4s. 6d. Find the prices of tea and coffee per lb.

7. The weight (w) of a given body in the neighbourhood of a heavenly body (such as the earth or the moon) is proportional directly to the mass (M) of the heavenly body and inversely to the square of the distance (d) of the given body from the centre of the heavenly body. Express this relation in a formula.

What value must be given to the constant in this formula so that the latter may give the weight of a body which weighs 1 lb. on the surface of the earth; the mass of the earth being taken as 1 and its radius as 4000 miles? (A thousand miles is to be taken as the unit of distance.)

What would this body weigh on the surface of the moon? (The moon's radius may be taken as 1080 miles, and its mass as $\frac{1}{103}$ of the earth's mass.)

8. In Jules Verne's "Journey to the Moon," bodies were found to lose their weight altogether at a certain point between the earth and the moon. Calculate (in thousands of miles) the distance of this point from the earth's centre, assuming the distance between the centres of the earth and moon to be 240,000 miles.

9. I have a wooden rectangular box 12 inches long, 9 inches wide and 5 inches deep. A sheet of glass is fixed inside it so that one edge lies along the bottom edge of one end of the box and the other along the top edge of the other end of the box. Find the angle between the plane of the glass and the bottom of the box. Find also the angle between a diagonal of the glass sheet and the bottom of the box.

10. The diameter of the circular tower of a castle is 40 feet. Starting from the wall the roof rises at an angle of $35\frac{1}{2}°$ with the horizontal plane until it encloses a circle 12 feet in diameter. It now rises at a steeper slope to a level 20 feet above the top of the wall. Finally it is capped by a cone 5 feet high with a semi-angle of 31°. Sketch a vertical section of the roof. Calculate (i) the slope of the middle section; (ii) the slope of the edges of the cone; (iii) the vertical height of the middle section; (iv) the diameter of the base of the cone; (v) the total vertical height of the roof.

D. Statistics.

1. The following numbers give the heights in inches of seven boys—measured in each case on the fourteenth birthday: 62, 58, 59, 63, 58, 59·5, 60·5.

Arrange the heights in a column in order of magnitude, and indicate the middle number of the series. This is called the **median.** Calculate also the arithmetic average or **mean** of the heights.

2. Set beside the column of No. 1 a second column showing how much greater or less each height is than the median height. Take the average of these differences. [Note that you must divide the sum by 7 not 6.] This average is called the **mean deviation from the median.**

3. Add a third column giving the difference between each height and the arithmetic mean of the heights. Obtain the average of the differences. This number is the **mean deviation from the mean.** Which of the two mean deviations is the less?

4. Find (*a*) the median, (*b*) the arithmetic mean, (*c*) the mean deviation from the median, (*d*) the mean deviation from the arithmetic mean in the case of each of the following sets of numbers:—

(i) 4, 13, 10, 6, 7;
(ii) 20, 16, 18, 9, 17, 11, 17, 8, 15;
(iii) 17, 14, 11, 6, 21, 17, 10, 12, 13, 12, 15, 14, 7.

Note.—The results of Nos. 2-4 illustrate the following facts: (i) the median may either be equal to, greater than, or less than the mean; (ii) whether the median is greater or less than the mean, the mean deviation from the median is always less than the mean deviation from the mean. It is, in fact, less than the mean deviation from *any* other of the measurements. For this reason when we speak of "the mean deviation" without further specification the mean deviation from the median is the number intended.

The terms "median" and "mean deviation" are often used to express the general result of a group of measurements. The median fixes the position of the middle of the group, the mean deviation measures the degree of **dispersion** of the measurements on either side of the middle one.

5. Two marksmen, A and B, fire five rounds each at a target. The following figures give the distance in inches of each "hit" from the centre of the bull's eye. Express numerically the value of each performance by means of the median and mean deviation. Which do you think is the better shot?

A—15, 8, 17·5, 10·5, 15;
B—10·5, 13·5, 16, 9, 6.

6. When there is an even number of measurements there is, strictly speaking, no median. The name is, however, generally given to the number half-way between the two middle measurements. What is the median of the following measurements? Prove *without calculation* that the mean deviation would be the same if 13 or 14 were assumed as the median. Calculate the mean deviation, using one of these numbers instead of the number which you give as the median:—

16, 12, 12, 14·5, 19, 22·5.

7. Find the arithmetic mean of the measurements given in No. 6 and the mean deviation from the mean. Is it greater or less than the mean deviation from the median?

8. Out of a certain form eleven pupils succeeded in solving one of the more difficult problems in this book. Their ages (in years and months) were:—

12 : 5, 12 : 8, 13, 13 : 2, 13 : 5, 13 : 8, 13 : 11, 13 : 11, 14 : 2, 14 : 4, 14 : 9.

Describe the general position and dispersion of this group.

Note.—In the foregoing group of measurements 13 (which is a quarter of the way up the series) and 14 : 2 (a quarter of the way down the series) are called the **lower** and **upper quartiles**. The difference between them (1 : 2) is called the **interquartile range**. The **semi-interquartile range**, here 0 : 7, is often taken, instead of the mean deviation, as a measure of the dispersion of the measurements. It may be called, more briefly but less accurately, the **quartile deviation**.

Observe that if there were 12 measurements the quartiles would be taken to lie half-way between the third and fourth terms from each end of the series.

9. Find the median, the quartiles, and the quartile deviation in the case of each of the following groups of measurements :—

(i) 3·8, 4·1, 4·1, 4·7, 5·2, 5·5, 6·3, 6·3, 6·9, 7, 7·2, 7·5, 7·7, 8·1 ;
(ii) 13·7, 14·2, 14·2, 15·6, 15·9, 16·3, 16·5, 17, 17·1, 17·4, 17·6, 18·2, 18·7, 18·8, 19·2, 20·1, 20·7, 21·2, 22·8, 23.

10. Two sets of measurements of a quantity were made, each set containing a dozen measurements. The median was in each case 24·8. The dispersion of the measurements, as measured by the mean deviation, was 4·8 for the one set and 2·3 for the other. Which set was the more trustworthy and why?

11. The medians of two sets of twenty measurements of a length were 30·7 cms. and 25·6 cms. respectively, while the quartile deviation was in each case 4·2 cms. Which was the better set of measurements?

12. Two surveyors, A and B, determined, each the same number of times, the height of a certain hill. The median of A's results was 824 feet, and the mean deviation 2·3 feet. The median of B's results was 822 feet and the mean deviation 2·1 feet. Which surveyor obtained the most consistent results?

13. Measurements of a number of wooden discs were made in order to find the ratio of the circumference of a circle to the diameter. The results are shown in the table. Plot on

squared paper the points which represent the various measurements. Let P be any one of these points, PN the perpendicular upon the horizontal axis, and O the origin. Then if PO be joined the value of the ratio given by the corresponding pair of measurements is PN/NO—i.e. tan PON. The ratios given by the other pairs are obtained similarly by joining the corresponding points to O. The fan of lines has a median and quartiles which correspond to the median and quartile values of the ratio. The best way to describe the general result of the measurements will be to state the median and the quartile deviation of the measured ratios.

In practice the median should be drawn as a thin but firm line, and the two quartiles as fine dotted lines. The median line is to be regarded as the "graph" of the measurements while the closeness of the quartiles serves as an ocular indication of its trustworthiness. The other lines need not be drawn.

Exhibit graphically by this method the general result of the measurements given in the table and calculate the median and quartile deviation of the ratios.

Diam.	2·7	3·5	4·0	4·5	5·0	5·8	6·0	6·6	7·3	7·5	8	cms.
Circf.	8·1	11·4	12·7	14·5	15·2	17·9	18·8	20·5	23·4	23·6	25·0	cms.

14. A variable P is known to be directly proportional to another variable Q, i.e. $P = kQ$, k being a constant. The following measurements were made in order to determine k. Exhibit graphically and express numerically the general result of the measurements.

Q	1·5	2·0	3·0	3·9	5·1	6·2	6·8	7·2	8·6	9·1	9·9	10·4	11·0	11·5	12
P	3·6	5·3	7·9	9·4	12·2	16·0	17·2	17·7	21·8	22·6	25·6	26·0	27·7	27·8	29·7

15. A variable P is known to be inversely proportional to another variable Q, i.e. $P = k/Q$, k being a constant. A number of corresponding measurements of the variables are given in the table. Plot the values of P against the *reciprocals* of the values of Q so as to exhibit graphically the various values of k given by the different measurements. What is their median and quartile deviation?

Q	1	2	3	4	5	6	7	8	9	10
P	1·9	0·9	0·67	0·49	0·46	0·37	0·31	0·27	0·18	0·17

16. Draw on a single sheet the inverse proportion curve corresponding to the median value of k in No. 15 and also the curves corresponding to the quartile values. Draw the median curve with a firm line and the quartile curves with dotted lines.

17. A variable P is known to vary directly as the square of a second variable Q. The table gives a series of corresponding measurements of the variables. Plot P to Q^2 in order to exhibit the values which the measurements give for k in the formula $P = kQ^2$. Calculate the median and the quartile values.

Q	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	6 0
P	0·75	0·95	1·8	2·91	3·78	6·62	8·27	10·42	12·5	14·23	19·78

18. Draw as in No. 16 the median and quartile curves which correspond directly to the measurements of P and Q given in the table of No. 17.

19. The relation between two variables is known to be given by the formula $P = k\sqrt{Q}$. Exhibit graphically the values of k derived from the following measurements, indicating as in No. 14 the median and quartile values.

Q	1·0	2·0	5·0	8·5	10·0	13·5	16·0	17·5	20·0	23·5	28·0	31·0	35·5	40
P	1·35	2·35	3·05	4·5	4·6	5·05	5·6	6·85	7·15	7·45	7·85	8·15	9·1	10·15

20. Draw as in No. 16 the median and quartile curves which correspond directly to the measurements given in No. 19.

E. Surds.

Note.—In carrying out calculations with surds it is generally best to follow the opposite plan to that of Exs. V and VI. Instead of replacing an expression by its factors the factors should be replaced by their product before the values of the surds are substituted. In this way troublesome multiplications and divisions are avoided.

Assume in these examples that $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$; $\sqrt{5} = 2{\cdot}236$, $\sqrt{6} = 2{\cdot}449$, $\sqrt{10} = 3{\cdot}162$.

1. Find to three decimal places the value of—

(i) $\sqrt{(3 - \sqrt{2})}\sqrt{2}$; (ii) $(\sqrt{3} - \sqrt{2})/\sqrt{2}$;
(iii) $(1 + \sqrt{5})/\sqrt{5}$; (iv) $(\sqrt{3} - 1)\sqrt{2}/\sqrt{3}$.

2. Find to three decimal places the value of—

(i) $(1 + \sqrt{2})(1 + 5\sqrt{3})$; (ii) $(7\sqrt{5} - 2)(\sqrt{2} - 1)$; (iii) $(2\sqrt{3} + \sqrt{5})(1 + 3\sqrt{2})$.

3. Evaluate—

(i) $(1 + \sqrt{2})^2$; (ii) $(2 - \sqrt{3})^2$; (iii) $(\sqrt{3} + \sqrt{2})^2$; (iv) $(3\sqrt{5} - 1)^2$; (v) $(2\sqrt{5} + 3\sqrt{2})^2$.

4. Evaluate—

(i) $(1 + \sqrt{2})^3$; (ii) $(2 - \sqrt{3})^3$; (iii) $(2\sqrt{3} - \sqrt{2})^3$.

5. Find the value of the following products :—

(i) $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$; (ii) $(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3})$;
(iii) $(\sqrt{17} - 4)(\sqrt{17} + 4)$; (iv) $(2\sqrt{5} + 1)(2\sqrt{5} - 1)$;
(v) $(7 + 3\sqrt{2})(7 - 3\sqrt{2})$; (vi) $(3\sqrt{5}) + (2\sqrt{8})(3\sqrt{5}) - (2\sqrt{8})$.

6. Complete the identities :—

(i) $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) =$;
(ii) $(2\sqrt{a} - 3\sqrt{b})(2\sqrt{a} + 3\sqrt{b}) =$;
(iii) $(\sqrt{ab} + 1)(\sqrt{ab} - 1) =$;
(iv) $(\sqrt{pq} - \sqrt{r})(\sqrt{pq} + \sqrt{r}) =$;
(v) $\left(\sqrt{\frac{p}{2}} - \sqrt{\frac{q}{3}}\right)\left(\sqrt{\frac{p}{2}} + \sqrt{\frac{q}{3}}\right) =$;
(vi) $\left(1 - \sqrt{\frac{ab}{5}}\right)\left(1 + \sqrt{\frac{ab}{5}}\right) =$;
(vii) $(a - \sqrt{b})(a + \sqrt{b}) =$;
(viii) $(a - 3\sqrt{b})(a + 3\sqrt{b}) =$.

Note.—The fraction

$$\frac{1}{\sqrt{5} - \sqrt{2}} = \frac{1}{\sqrt{5} - \sqrt{2}} \times \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}}$$
$$= \frac{\sqrt{5} + \sqrt{2}}{(\sqrt{5})^2 - (\sqrt{2})^2}$$
$$= \frac{\sqrt{5} + \sqrt{2}}{5 - 2}$$
$$= \tfrac{1}{3}(\sqrt{5} + \sqrt{2}).$$

This form of the fraction is much easier to evaluate than the original form. The expressions $(\sqrt{5} - \sqrt{2})$ and $(\sqrt{5} + \sqrt{2})$ are said to be **conjugate binomial surds.** We have, therefore, the rule that the binomial surd in the denominator of a fraction may be removed by multiplying both numerator and denominator by the conjugate surd.

7. Convert the following fractions into the form most suitable for evaluation :—

(i) $\dfrac{1}{\sqrt{3}-\sqrt{2}}$; (ii) $\dfrac{1}{\sqrt{3}+\sqrt{2}}$; (iii) $\dfrac{1}{3-\sqrt{5}}$;

(iv) $\dfrac{1}{\sqrt{7}-2}$; (v) $\dfrac{1}{2\sqrt{3}+1}$; (vi) $\dfrac{1}{3\sqrt{5}-1}$;

(vii) $\dfrac{2}{6\sqrt{11}-\sqrt{5}}$; (viii) $\dfrac{4\sqrt{7}}{\sqrt{7}+\sqrt{5}}$; (ix) $\dfrac{2\sqrt{3}}{5\sqrt{11}-4\sqrt{3}}$.

8. Simplify the following practical expressions, giving the final denominator in a rational form :—

(i) $\dfrac{1}{a}-\dfrac{1}{a+\sqrt{2}}$; (ii) $\dfrac{1}{p\sqrt{3}-4}-\dfrac{1}{p\sqrt{3}}$;

(iii) $\dfrac{1}{a\sqrt{5}}-\dfrac{1}{a\sqrt{5}+b}$; (iv) $\dfrac{\sqrt{2}}{\sqrt{3}-p}-\dfrac{\sqrt{2}}{\sqrt{3}}$.

9. Find values of n which accord with the following statements :—

(i) $(n-1)\sqrt{2}=\sqrt{5}+2$; (ii) $n(1+\sqrt{3})=2n+4\sqrt{5}$;
(iii) $n^2-6n=1$; (iv) $n^2-n-1=0$;
(v) $\sqrt{(2n-1)}=\sqrt{5}-2$.

10. A straight line AB at length a is divided at H so that $(AH)^2 = AB.BH$. Show that

$$AH = \frac{a}{2}(\sqrt{5}-1).$$

Calculate the position of H when the line is 20 cms. long. Verify the correctness of your calculation.

F. Test Paper 4.

1. If the course of a river is obstructed (e.g. by a sudden narrowing of its bed as at "the Strid," near Bolton Abbey) the level of the water in front of the obstruction is higher than it would otherwise have been. The rise in feet is given by the formula :—

$$R = \left(\frac{v^2}{58\cdot6} + 0\cdot05\right)(p^2 - 1).$$

$v \equiv$ velocity of the river (in feet per second) before the obstruction is reached; $p \equiv$ cross-section of river before the obstruction divided by cross-section at obstruction.

(i) A river flowing with a velocity of 4 feet per second has suddenly to pass between rocks where the cross-section of the bed is reduced to $\frac{1}{4}$ of its amount just before the obstruction. How much will the water rise at this point?

(ii) Repeat, substituting 6 feet per second and $\frac{2}{5}$ for the numbers given.

2. Write down a formula for calculating the normal speed of a river from the rise of level caused by a sudden constriction of its bed. (See No. 1.)

3. Take any number of three different digits, a, b, c. Obtain a second number by reversing the order of the digits. Find the digits in the difference between the two numbers. Obtain a third number by reversing the order of the digits in the difference. Add this third number to the difference between the first two. Show that the sum will always be the same.

4. The tangent of a certain angle is $\dfrac{2n+1}{2n(n+1)}$. Find an expression (i) for the sine, (ii) for the cosine of the same angle.

5. Show that $a^4+a^2b^2+b^4=(a^2+b^2)^2-a^2b^2$. Hence find the factors of the former expression.

6. (i) Simplify the fractional expression :—

$$\frac{7}{(a-1)(2a-3)}-\frac{6a-5}{2a-3}+\frac{3a+4}{a-1}.$$

Use your result to find (ii) the value of the expression when $a=21\cdot5$; (iii) the value of a for which the expression has the value 2.

7. When 3 is added to the numerator and also to the denominator of a certain fraction its value becomes 0·7. When the 3 is subtracted its value is 0·25. Find the fraction.

8. What can you deduce from the following statements about the value of n in each case?

(i) $7n-3>4(n+1)$;

(ii) $(5n+3)^2-(3n-5)^2>(4n-1)^2$;

(iii) $\dfrac{1}{2n+1}<3+\dfrac{1}{2n-1}$.

9. A certain number of persons promised to subscribe equally to a gift which cost £20. Three of them failed to pay their subscriptions. Each of the others had, therefore, to increase his subscription by 18s. How many persons originally undertook to subscribe, and what was the amount which they promised?

10. Where I stand the top of a tree is seen to have an altitude of 11°. I walk 100 yards towards it along a slope

which rises at an angle of 10° with the horizontal. The altitude of the top of the tree is now 12°. Calculate the vertical distance of the tree-top above the level of the eye, and the horizontal distance between the vertical through the tree-top and the vertical through the eye.

G. Test Paper 5.

1. The rate of movement of coal-gas is described by the formula :—

$$Q = 1000\sqrt{\frac{Hd^5}{gl}}.$$

$Q \equiv$ quantity of gas in cubic feet per hour; $l \equiv$ length of pipe in *yards*; $d \equiv$ diameter of pipe in inches; $g \equiv$ the specific gravity of the gas, i.e. the weight of any bulk of it divided by the weight of an equal bulk of air; $H \equiv$ pressure of gas supply, measured by the water gauge as a "head" of water (in inches).

(i) At what rate can gas be supplied by a 3-inch main, 300 yards long under a pressure of $\frac{3}{4}$ inch of water? (The specific gravity of the gas may be taken as 0·45.)

(ii) The same, the pipe being 1000 yards long and 4 inches in diameter, and the head being 0·8 inch of water.

2. From the formula of No. 1 obtain formulæ for finding (i) the pressure of the gas supply, (ii) the diameter of the supply-pipe.

[The fifth root of 0·1 may be taken as 0·63.)

3. A and B are to choose any two numbers with the condition that one is to be even and the other odd. A's number is multiplied by 2 and B's by 3 and the products are added. Show that if A's choice was an even number the sum will be even, if an odd number the sum will be odd.

Show also that these consequences will follow if A's choice be multiplied by any even number and B's by any odd number.

4. Find the factors of

(i) $16a^4 + 36a^2b^2 + b^4$;

(ii) $a^2 + 1 + \frac{1}{a^2}$; (iii) $p^2 - q^2 + 2q - 1$.

5. Given that $m = 2\sqrt{3} - 1$ and $n = 2\sqrt{3} + 1$, find the value of

$$\text{(i)}\ \frac{1}{m^2} + \frac{1}{n^2}; \qquad \text{(ii)}\ \frac{1}{m^2} - \frac{1}{n^2}.$$

6. Show that the expressions $\dfrac{a^3 - b^3}{a - b} + \dfrac{a^3 + b^3}{a + b}$ and

$$\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{a^3 - b^3}{a - b} - \frac{a^3 + b^3}{a + b}\right)$$

always have the same value. Find their value when

$$a = \sqrt{2} + 1 \text{ and } b = \sqrt{2} - 1.$$

7. Find values of n which satisfy the following conditions:—

(i) $\frac{7}{6}n - \frac{1}{4} = \frac{11}{36}(3n + 1) + \frac{17}{18}$;

(ii) $\dfrac{5}{\frac{1}{3} - n} - \dfrac{13}{1 - 3n} = 4$;

(iii) $\left(5 + \dfrac{n}{2}\right)\left(5 - \dfrac{n}{2}\right) + \dfrac{n^2}{4} = n + 12$;

(iv) $\dfrac{n - a}{1 - a} - \dfrac{n - 1}{1 + a} - n = 0$;

(v) $(n - 1)^2 + (n + 1)^2 + (2n + 3)^2 = 29$.

8. I have a large number of match boxes. I wish to set them side by side on a table so that they shall form a rectangle in which the number of rows of boxes is the same as the number of boxes in a row. The first time I arrange them in this way there are 24 boxes over. I try to arrange them, therefore, with one more box in a row. This time the number is 29 short of that required. How many boxes are there?

9. Two small bodies are moving along the same straight line and are 3000 feet apart. Their velocities are such that if they were going the same way the faster would catch the slower up in thirty seconds, while if they were moving towards one another they would meet in twenty-five seconds. Find the velocities.

10. A hollow cone of paper rests on a geographical globe in such a position that it touches the globe along the 50th parallel of latitude. The radius of the globe is 20 cms. Calculate (i) the semi-angle of the cone; (ii) the distance between the apex of the cone and the centre of the sphere.

H. Test Paper 6.

1. In a water supply a certain "head" is necessary merely to overcome the friction of the pipes (i.e. if there were no friction the water would be delivered more quickly). The head (in feet) used up in this way is given by the formula :—

$$h = \frac{l}{d}\left(0{\cdot}0036 + \frac{0{\cdot}0043}{\sqrt{s}}\right)\frac{s^2}{32}.$$

$s \equiv$ speed of the water-flow in feet per second; $l \equiv$ length of pipe in feet; $d \equiv$ diameter of pipe in inches.

(i) Calculate the head required to overcome the friction in a 2-inch pipe 1000 feet long in which the water is running 4 feet per second.

(ii) The same, the length being a mile and a half, the diameter of the main 6 inches, and the speed of the water 9 feet per second.

2. The following formulæ are copied from an electrical engineer's pocket-book. Obtain from the first of them a formula for B and from the second a formula for G.

(Does your ignorance of the meaning of the formulæ cause any difficulty ?)

$$\text{(i)}\quad \mathrm{E} = \mathrm{d}\left(\mathrm{B} + \frac{\mathrm{Gs}}{\mathrm{G} + \mathrm{s}} + \mathrm{R}\right);$$

$$\text{(ii)}\quad \mathrm{R} = \left(\mathrm{r} + \frac{\mathrm{Gs}}{\mathrm{G} + \mathrm{s}} + \mathrm{B}\right).$$

3. A person, A, having taken any number he pleases out of a heap of counters, another person, B, is told to take p times as many. The person who conducts the game specifies p but does not know how many counters A took. A is now told to hand to B a certain specified number, q, of the counters which he holds, and B is told to give in exchange to A p times as many counters as A has left. Show that B will have at the end $(p + 1)q$ counters. Give a numerical illustration.

4. The sine of a certain angle is $4n^2 + 1$. Find expressions for (i) the cosine, (ii) the tangent of the same angle.

5. For what value of p does the expression

$$\frac{p^2 + 2p - 3}{p^2 - 2p - 8} \div \frac{p^2 + p - 6}{p^2 - 3p - 4} = 5?$$

6. Find values of n which comply with the following conditions :—

(i) $$\frac{25 - \frac{n}{3}}{n + 1} + \frac{16n + 4\frac{1}{3}}{3n + 2} = \frac{23}{n + 1} + 5;$$

(ii) $$c^2 = \frac{b^3 - c^3}{b - c} - \frac{bc(b + c)}{n};$$

(iii) $n^2 + 10n + 3 = 2n^2 - 5n + 53$;

(iv) $$\frac{(2n + 5)n}{2(n - 1)(n + 2)} = \frac{6}{5}.$$

7. A circular disc is to be made by cutting two sectors from sheets of gold and of lead of the same thickness and putting them together. How many degrees will there be in each sector if 19 sq. cms. of the gold plate weigh as much as 11 sq. cms. of the lead plate?

8. The position in which the lens of a magic lantern must be fixed so as to throw on the screen a clear image of the slide is given by the formula $(x - a)(y - b) = c$.

In this formula $x \equiv$ the distance from the front lens to the screen, $y \equiv$ the distance from the same point to the slide, while a, b, and c are constants. Find the value of the constants in the case of a certain lantern from the following data:—

Distance of screen: 3 feet 3 inches; 5 feet 11 inches; 11 feet 3 inches.

Distance of slide: 14 inches; 13 inches; $12\frac{1}{2}$ inches.

What should be the distance of the front of the lens from the slide when its distance from the screen is 16 feet 7 inches?

9. Write down a formula for calculating, to a first approximation, the volume of a sphere whose radius is 1·01 r, when the volume of a sphere of radius r is already known.

An error of 1 per cent is made in measuring the radius of a sphere whose radius is actually 5 inches. What will be, to a first approximation, the error in the calculation of the volume?

[Assume $V = \frac{4}{3}\pi r^3$ where $\pi = \frac{22}{7}$.]

10. Find the values of the angle α which satisfy the following conditions:—

(i) $\sin\alpha - (4 - 2\sin\alpha) = 7(\sin\alpha - 1)$;

(ii) $(\tan\alpha - 1)(\tan\alpha - 2) = (\tan\alpha - 3)(\tan\alpha - 4)$;

(iii) $9\sin^2\alpha - 24\sin\alpha\cos\alpha + 16\cos^2\alpha = 0$;

[Divide by $\cos^2\alpha$.]

(iv) $$\tan\alpha + \frac{1}{\tan\alpha} = \frac{5}{2}.$$

SECTION II.

DIRECTED NUMBERS.

EXERCISE XXVII.

THE USE OF DIRECTED NUMBERS.

A.

Use directed numbers to solve the following problems :—

1. A lift in a large hotel starts at ten o'clock on the fifth floor. During the next hour it makes the following journeys : Up three floors, down five, down two, up four, down one, up five, down eight, up six, up one, down three, down two. Where is it at the end of the hour?

2. On January 18, 1907, the highest temperature during the day was 37·1°. On successive days after that date the highest reading of the thermometer went down 2·9°, up 5·9°, down 7·2°, down 8°, down 1·3°, up 3·4°, up 3·4°, up 6·3°, up 9·4°. What was the highest temperature (i) on the 24th ; (ii) on the 27th?

3. The following table gives the rainfall in London during each of the months of 1907. The numbers in the second row are the average amounts of rainfall for each month recorded during the preceding fifty years. In a third row put a directed number to show the difference between the numbers for 1907 and the averages. Was the total fall for 1907 above or below the average?

Jan.	Feb.	Mar.	Apl.	May	June
1·09	1·27	0·9	3·48	1·46	2·64
1·18	1·48	1·46	1·66	2·00	2·02

July	Aug.	Sep.	Oct.	Nov.	Dec.
0·96	2·33	0·63	3·44	4·13	2·74
2·47	2·35	2·25	2·81	2·27	2·13

4. A boy puts 10s. into the Post Office Savings Bank on January 2 and 3s. on February 18 ; he draws 15s. on March 23, puts in 25s. on his birthday (April 19), draws 4s. on June 15, 6s. on August 4, puts in 5s. on October 10, and draws two guineas on December 18. During the year his deposit earned

7s. 8d. If his balance at the beginning of the year was £16 4s. 7d., what was it at the end of the year?

5. The following table gives the time by the clock when the sundial registers noonday at different periods of the year. Make a table of directed numbers that will show the correction necessary to turn sundial-time into clock-time at those periods.

Jan.	Feb.	Mar.	Apl.	May	June
1 12.3	1 12.13½	1 12.12½	1 12.4	1 11.57	1 11.57½
15 12.9	14 12.14½	15 12.9	15 12.0	15 11.56	15 12.0

July	Aug.	Sep.	Oct.	Nov.	Dec.
1 12.3½	1 12.6	1 12.0	1 11.50	1 11.43	1 11.49
15 12.6	15 12.4½	15 11.55	15 11.45	15 11.44	15 11.55

6. Make up two examples like any two of the foregoing. Work them both out.

B.

7. Find the total value of each of the following sets of directed numbers:—

(i) $+ 7 + 12 - 9 - 8 + 17 - 24$.
(ii) $- 3{\cdot}2 - 4{\cdot}7 + 2{\cdot}8 - 0{\cdot}6 - 12 + 9{\cdot}7 - 2{\cdot}3$.
(iii) $+ 0{\cdot}7 - 23 - 1{\cdot}8 + 73 - 0{\cdot}03 + 5{\cdot}43 + 81{\cdot}5 - 143{\cdot}72$.
(iv) $-$ £3 14s. $+$ £6 7s. 5d. $+$ £3 0s. 7d. $-$ 16s. 8d. $-$ £7 12s. 4d.
(v) $+ 7\frac{1}{2}$ lb. $- 4\frac{7}{8}$ lb. $- \frac{3}{8}$ lb. $+ 6\frac{3}{4}$ lb. $- 12$ lb. $+ 7\frac{5}{16}$ lb.
(vi) $-$ 2 hrs. 14 mins. $+$ 3 hrs. 7 mins. $-$ 5 hrs. 37 mins. $-$ 43 mins. $+$ 1 hr. $2\frac{1}{2}$ mins.
(vii) $64° - 2{\cdot}3° - 0{\cdot}6° + 4{\cdot}1° - 2° + 5{\cdot}05°$.

8. Make up problems to fit one of the foregoing sets of numbers.

Note.—The numbers 38 and 26 can be written $32 + 6$ and $32 - 6$ respectively or, more concisely, 32 ± 6. The number 32 is in this case called the **arithmetical mean** of 38 and 26.

9. Express each of the following pairs of numbers in terms of its mean, as in the foregoing example.

(i) 27 and 13; (ii) 94 and 66; (iii) 81 and 38; (iv) 12·6 and 7·7; (v) 0·43 and 2·05.

10. (i) £136 12s. and £79 6s.; (ii) £23 7s. and £64 13s.; (iii) £8 15s. and £1 14s. 8d.; (iv) 2 cwt. 17 lb. and 4 cwt. 5 lb.; (v) 14 hrs. 45 mins. and 11 hrs. 9 mins.

11. The first of each of the following pairs of numbers is the mean between the second and another number not given. Find in each case the other number.

(i) 31 and 18; (ii) 4·7 and 6·8; (iii) £154 6s. and £137 15s.; (iv) $14\frac{1}{2}$ cubic feet and $21\frac{3}{4}$ cubic feet; (v) 59·3° and 53·2°.

12. Use a diagram to find the mean of the following pairs of directed numbers. When you have discovered the rule for calculating the mean, you can use it instead of the diagram.

(i) + 7 and − 17; (ii) + 12 and − 6; (iii) + 4 and − 10; (iv) + 16·7 and + 14·9; (v) − 13 and − 27; (vi) − 7·3 and − 19·4; (vii) + 3·8 and − 46·3.

13. The first of each of the following pairs of directed numbers is the mean between the second number and another number not given. Find the other number in each case, using a diagram to obtain or to check your results.

(i) − 6 and − 16; (ii) + 3 and − 41; (iii) + 4·8 and − 17·4; (iv) − 5·3 and + 1·6; (v) − 5·8 and + 14·7; (vi) − 41 and − 53; (vii) − 7·5 and − 8·2.

C.

Note.—Suppose that the maximum (*i.e.* the highest) temperature on a certain day was 70° F. and the minimum (*i.e.* the lowest) temperature was 50°, so that the mean temperature was 60°. We can conveniently express these facts by writing that the extreme temperatures were 60° ± 10° (which is read "60° *plus* or *minus* 10°").

14. The coldest weeks of 1907 were the fourth week in January and the second in February; the hottest was the third in July. The following table gives the maximum and minimum readings of the thermometer for each day in those weeks. Express the daily temperature in terms of the mean daily temperature as in the above example.

Jan.	Max. °	Min. °	Feb.	Max. °	Min. °	July	Max. °	Min. °
20	38·1	31·7	3	32·9	29·9	14	71·9	57·3
21	44·0	34·2	4	33·8	30·1	15	78·5	53·1
22	36·8	26·8	5	36·0	31·0	16	77·0	54·6

Jan.	Max. °	Min. °	Feb.	Max. °	Min. °	July	Max. °	Min. °
23	28·8	23·6	6	39·0	30·3	17	72·1	53·5
24	27·2	22·4	7	38·8	23·5	18	75·8	52·6
25	30·7	24·1	8	38·2	23·6	19	78·9	48·7
26	34·1	23·3	9	41·2	27·0	20	73·3	50·9

What facts about the temperatures are brought out by this way of writing them?

15. During a week in June I found the temperature in a shady corner of my garden daily at ten o'clock a.m. The readings were 64°, 67°, 62°, 51°, 53°, 57°, and 52° respectively. Looking at these numbers I guessed that the average temperature was 58°. Express the given temperatures in terms of this guessed (or "trial") average, and use the results as a quick way of finding out whether I was right.

16. During another week the temperatures were 54°, 50°, 48°, 43°, 40°, 45° and 49°. I guessed the average or mean temperature for the week to be 48°. Was I right?

17. During yet another week the temperatures were 49°, 48°, 53°, 51°, 58°, 56°, 49°. What do you guess the average to be? Test your guess. If you were wrong use your test to find the correct mean in the quickest way.

18. The following table gives the times of departure and arrival of trains from London (Paddington) to Gloucester.

Calculate how many minutes each train takes, and determine by the use of a "trial average" the mean length of the journey:—

Depart	1.0	5.40	7.30	9.0	10.50	11.40	3.15	4.45	5.15
Arrive	3.56	9.13	10.30	12.4	1.43	3.2	5.54	7.59	8.40
Depart	6.10	9.15							
Arrive	8.50	12.26							

19. Find by use of a trial average the mean of the following numbers:—

(i) 115, 112, 104, 101, 109, 87, 79, 93.
(ii) 16·5, 16·1, 18·9, 14·7, 13·1, 12·5.
(iii) $+11$, $+7$, $+19$, $+9$, -3, -7, -1.
(iv) $+4$, $+3$, $+2$, 0, -5, -11, -9, -8.

20. Use the results of your answer to No. 14 to find the mean temperature during each of the three weeks mentioned in that question. Which was the coolest week in 1907?

21. Find the mean daily variation in temperature during each of the same weeks.

22. Draw graphs to show the variation of the daily mean temperature about the weekly mean in each of these weeks.

23. In January, 1907, the moon was new on the 3rd, and full on the 18th. The following table gives the time at which it was south during this half of its monthly period. Find the mean value of the time interval between one "southing" and the next.

Day.	*Moon Sth.*	*Day.*	*Moon Sth.*
3	11.38 a.m.	12	7.43 p.m.
4	—	13	8.30
5	1.47 p.m.	14	9.17
6	2.47	15	10.6
7	3.43	16	10.55
8	4.35	17	11.44
9	5.24	18	—
10	6.11	19	12.32 a.m.
11	6.57		

24. Draw a graph to show the variation between the successive intervals and the mean interval in the last question.

25. The following table gives the times of high tide at London Bridge on the days mentioned in No. 23. Find (to the nearest minute) the mean interval between one high tide and the next. Compare the result with that of No. 23.

Day.	*Morn.*	*Aft.*	*Day.*	*Morn.*	*Aft.*	*Day.*	*Morn.*	*Aft.*
3	12.53	1.18	9	5.52	6.17	15	11.59	—
4	1.44	2.9	10	6.44	7.9	16	12.27	12.53
5	2.33	2.58	11	7.35	8.3	17	1.16	1.39
6	3.23	3.47	12	8.32	9.7	18	1.59	2.19
7	4.13	4.38	13	9.42	10.17			
8	5.2	5.26	14	10.53	11.27			

26. Add to the graph which you drew in answer to No. 24 a graph showing the variation of the tidal interval from its mean value. (It is convenient to draw the second graph in red ink or with dotted lines.)

27. The leader of a squad in "figure-marching" takes the following movements in succession: 21 paces north, 17 paces east, 19 paces south, 14 paces west, 11 paces north, 3 paces west. Draw a diagram to show his final position and then show how you could have determined it by calculation.

28. The following is the description of the movements made by a boy who is trying to get out of a maze: 35 yards N., 52 E., 36 N., 21 W., 18 S., 13 W., 18 N., 26 W., 44 S., 12 W., 32 N*., 41 W., 47 S., 30 E., 10 N., 20 W., 28 N., 28 E.,

53 S., 23 E. Where will he be after the movement marked "*" and at the end? Check your calculation by a diagram of the maze.

29. A dirigible balloon begins a testing flight by rising vertically through 120 feet. It subsequently carries out the following movements: 1560 feet N., 780 feet W., 1470 feet S., 840 feet E., 90 feet upwards, 900 feet E., 650 feet S., 140 feet downwards, 700 feet W., 640 feet N. Where will it now be with regard to its original position?

30. Invent another problem like No. 15 and solve it.

31. Invent another problem like No. 16 and solve it.

EXERCISE XXVIII.

ALGEBRAIC ADDITION AND SUBTRACTION.

A.

1. The following statements have reference to the movements of a lift in a large hotel, a movement from one floor to the next being taken as unit. The symbol n stands for "number of floors". State in words what problem is meant in each case and solve it.

(i) $n = (+2) + (+3)$.
(ii) $n = (+2) - (-3)$.
(iii) $n = (-5) + (+1)$.
(iv) $n = (-5) - (-1)$.
(v) $n = (+3) + (-5)$.
(vi) $n = (+3) - (+5)$.
(vii) $n = (-4) + (+4)$.
(viii) $n = (-4) - (+4)$.
(ix) $n = (-3) + (-5)$.
(x) $n = (-3) - (-5)$.
(xi) $n = (+4) + (-5)$.
(xii) $n = (+4) - (+5) - (-3)$.
(xiii) $n = (-7) + (+3) + (-1)$.
(xiv) $n = (-7) - (+3) - (-1)$.
(xv) $n = (-2) + (+3) - (-4)$.
(xvi) $n = (-2) - (-3) + (+4)$.

Note.—Examples Nos. 2-9 are to be set down in such a way as to show whether the problem is one of calculating an unknown resultant from given components, or of calculating an unknown component, given the resultant and the other component (or components). Use the symbol R for an unknown resultant and the symbol p for an unknown component.

2. A lift starts with a passenger from the ground floor of a hotel and carries him up to the seventh floor. There he finds that his room is three floors lower down, so the lift descends with him to that level. On which floor is his room?

3. A visitor in an hotel left his room and entered the lift in order to descend to the dining-room on the ground floor. He thought, of course, that the lift was about to go down, but it was actually going up and carried him up three floors and then descended seven floors to the dining-room level. On which floor was his room?

4. A motorist is about to start from the village of X in order to go northwards to Y. Noting that his supply of petrol is insufficient he drives south 5 miles to the nearest town Z, obtains additional petrol and then travels 52 miles to reach Y through X. What is the direct distance from X to Y?

5. A motorist starts from X to go to Y. Twelve miles from X he overtakes a friend who is cycling to a farm-house some distance beyond Y. He takes his friend and the bicycle into the motor-car and the two drive on for 38 miles till they reach the village of Z. On making inquiries here they find that they passed the farm-house 4 miles before entering Z. They drive back to it and the cyclist leaves the car. The motorist now has to drive 6 miles back to Y. How far is it from X to Y?

6. A cyclist starts for a distant place. Five miles from home he has to stop to mend a puncture. Having ridden 11 miles farther on he discovers that he has missed a turning which he should have taken. He rides back 3 miles to the sign-post and finds that he has now 4 miles to go to his destination. How long would the ride have been if he had not made the mistake?

7. A party of people start from a seaside town A to sail to another town B, 9 miles away. The tide carries their boat past this town to a place, C, 3 miles farther on. They effect a landing here and sail back to their original destination when the tide turns. How many miles did they sail from A to C?

8. A cyclist started for a place 19 miles away. He stopped to buy a repair outfit at a town 6 miles from home and stopped again for tea 9 miles farther on. He was told here that he ought to have taken a turning some distance back. He rode back to this turning and found that he has still 7 miles from his destination. How many miles had he gone beyond the turning?

9. A cyclist starting from A to ride to B was told that the distance was 14 miles, and that he must look out for a narrow

lane on the right which he was to follow. After some time he made inquiries and found that he had ridden 2 miles beyond the turning. He went back, found the lane, and after 6 miles reached B. How far is it from A to the beginning of the lane?

10. Explain the meaning of the following problems and solve them, given $a = -3$, $b = +5$, $c = -7$:—

(i) $R = a + b$.	(ii) $R = a + 2b$.
(iii) $R = 3a + 2b + c$.	(iv) $d = a - c$.
(v) $d = a + b - c$.	(vi) $d = a - b + c$.
(vii) $k = 4a - 2b - 3c$.	(viii) $m = b + 3c - 6a$.

B.

11. A clerk occupies a post in which his salary increases £7 10s. per annum. His present salary is £150. Write down a formula which will give his salary *t* years hence.

How could you use the same formula to calculate his salary *t* years ago? Use your formula to find (i) what his salary will be in four years' time, and (ii) what it was six years ago.

12. A man inherited a sum of money many years ago, but he has been withdrawing £12 a year from it to pay his life insurance premiums. Its present amount is £243. Write down a formula by which the amount at other times may be calculated (A, *t*).

Use your formula to find (i) how much he had fourteen years ago, (ii) how much will be left in twenty years' time if he lives so long.

13. A butcher owns two shops. The receipts from one are improving at the rate of £14 a month, from the other they are falling off at the rate of £19 a month. This month the two shops yielded together £1317. Find, by the use of a single formula his whole weekly takings (i) twelve months ago, (ii) eight months hence (T, *t*).

14. A liner crossing the Atlantic consumes 4 tons of coal every hour. When a certain distance out from Liverpool she has 620 tons in her bunkers. Write down a formula by which the amount of coal in her bunkers may be calculated for every hour after or before this moment (C, *t*).

How much coal (i) will she have in two days' time, (ii) had she two days ago?

15. Draw a graph to illustrate the preceding question.

How will you arrange your graph so as to distinguish times before the present moment from times after?

[It will be convenient to take 20 hours to the inch for the time-scale and 200 tons to the inch for the coal-scale.]

Use the graph to check the calculations you made with the formula and also to answer the following questions:—

(i) How much coal will the liner burn in 40 hours?
(ii) When will she have 500 tons left?
(iii) When did she have 800 tons?
(iv) She had 812 tons when she left Liverpool. How long has she been out?
(v) She will reach New York with 236 tons. How long does the crossing take her?

16. A cricket club begins the year with a balance of £33, but the subscriptions have for several years been £7 less than the annual expenses. Give a formula for the financial position of the club at the beginning of each year, and use it to find (i) what the position was four years ago, and (ii) what it will be in six years' time if the present conditions continue (B, t).

17. A rival club to the one in the last question begins the year with a deficit of £8, but its subscriptions have for several years exceeded the year's expenses by £4. Determine, by the use of a formula, the financial position of the club (i) five years ago, (ii) five years hence if the present conditions continue.

18. A baker who opened a new shop some years ago estimates that it has returned him his capital and, in addition, has yielded a total profit of £144. The receipts have always exceeded the expenses by £16 a month. Write a formula showing the financial position of the enterprise at the end of each month. What was it (i) two years ago, (ii) what will it be in four and a half months (P, t)?

19. The same baker estimates that another shop has still to bring him in £120 profit before it will have returned the whole of the capital originally invested in it, but that the receipts exceed the expenses at the rate of £24 a month. Write a formula showing the financial position of the shop at the end of each month. (i) What was it a year ago? (ii) What will it be in three years' time?

20. Illustrate the last two questions by drawing *on the same piece of squared paper* two graphs, one for each of the

shops. Use the graphs to find (i) when each of the shops had just paid, or will just have paid back the whole of the capital expended on it, (ii) when they will both have earned the same total profits.

Check each of these results by the formulæ.

21. The shop of No. 18 was opened three years ago, that of No. 19 two and a half years ago. Find by the graphs how much capital was invested in each. Check the results by the formulæ.

22. Why is it impossible to illustrate Nos. 16 and 17 by graphs similar to those just employed? What kind of graphs would be suitable?

23. A railway rises for several miles at a regular rate of 6 feet for every 100 feet of the slope. Just outside a certain station it is 300 feet above sea-level. Use a formula to find its height above sea-level (i) 750 feet farther up the slope, (ii) 450 feet farther down (h, d).

24. A long gallery in a coal mine beneath the sea slopes downwards from the foot of the shaft at a uniform rate of 1 in 7. When a certain side gallery branches off it is exactly 1000 feet beneath the sea. Write a formula that shall give the depth (D) of points at a given distance (d) from this place. Use it to find the depth at a place (i) 490 yards before you reach the place in question from the shaft, (ii) 1400 yards farther on.

25. It is said that after a certain depth the temperature of the earth beneath England increases 1° F. for every 50 feet that you descend towards the centre of the earth. At a depth of 3000 feet it is 110°. Write a single formula which shall give the temperature (T) at a point distant d feet below or above this point.

Use the formula to calculate the temperature at a depth of (i) 4000 feet, (ii) 1800 feet *below the surface.*

26. Write another formula expressing the temperature at a depth d below the surface.

Use this formula to calculate the temperature at the depths mentioned in the last question. Do your results agree?

27. Write a third formula by which the depth from the surface (d) can be calculated when the temperature (T) is given.

Use it to calculate the depth where the temperature is (i) 160°, (ii) 85°.

28. Draw a graph corresponding to the formula of No. 27. Show how you can use it to answer the questions of No. 25.

How would the graphs corresponding to the formulæ of Nos. 25 and 26 differ from this one?

29. It may be taken that the barometer falls practically 0·11 inch for every 100 feet you rise so long as the total ascent does not much exceed 2000 feet. At a village on the side of a Welsh mountain 750 feet above the sea the barometer stands to-day at 29·8 inches. Write a formula giving the heights of the barometer (B) at places on the mountain d feet higher or lower than the village.

Use the formula to calculate the height of the barometer (i) at the sea-level, (ii) at the top of the mountain which is 1820 feet high.

30. Write a formula giving the height of the barometer (B) at different heights (h) above sea-level.

Calculate (i) and (ii) of the preceding question by this formula.

31. Write a formula giving the height above sea-level (h) corresponding to a given height of the barometer (B). Use it to find the height where the barometer records (i) 29·1 inches, (ii) 30·3 inches.

C.

32. Find the value of P as given by each of the following formulæ: (1) by direct substitution; (2) by first simplifying the formula and substituting afterwards. Take $a = -2$, $b = +5$, $c = -7$. The meaning of the original formula is to be explained in each case and also the meaning of the simpler formula to which it is reduced:—

(i) $P = (a - 2b) + (3a - b)$.
(ii) $P = (3a + b) + (2b - a)$.
(iii) $P = (a - 4b) - (2a + 6b)$.
(iv) $P = (5b - 2a) - (a - 6b)$.
(v) $P = (a + b) + (2a - b + c)$.
(vi) $P = (3a - 2b + 3c) - (4a + 2b + 3c)$.

33. Reduce each of the following formulæ to the form most suitable for substitution. Find the value of the subject when $p = +10$, $q = -6$, $r = -2$:—

(i) $A = (2p - q + r) - (p - 3q + 4r)$.
(ii) $C = 2(3p + q) + 4(p - 2q)$.
(iii) $n = 3(p + q + r) - 2(p - q) - 3(q - r) - 4(r - p)$.

(iv) $E = \frac{1}{3}(2p - q) - \frac{1}{4}(3p - 2q)$.
(v) $B = \frac{2}{5}(3p - 4q) + \frac{1}{2}(4q - 5r) - \frac{3}{10}(5r - 6p)$.
(vi) $M = 1{\cdot}2(p + q - 2r) + 2{\cdot}3(p - 2q + r) - 3{\cdot}4(q + r - 2p)$.

34. Reduce the following expressions to the forms most suitable for substitution :—

(i) $\{2m + n - 3(m - n)\} + \{4(m + 2n) - 3(m - 2n)\}$.
(ii) $\{2{\cdot}5(a - 2b) + 3{\cdot}7(2a - b)\} - \{2{\cdot}1(a + 2b) - 0{\cdot}6(2a + b)\}$.
(iii) $\{2(1{\cdot}3p + 2) - 3(2{\cdot}7q - 1)\} - \{4{\cdot}2(p - q + 1) + 2{\cdot}8\}$.
(iv) $2\{4(\frac{2}{3}a - \frac{1}{4}b) + 5(\frac{3}{4}b - \frac{2}{5}c) + 6(\frac{4}{5}c - \frac{1}{2}a)\}$.
(v) $3\{5(\overline{3a - 5b} - \overline{a - 2b}) - 4(\overline{5a - 3b} - \overline{2a - b})\}$.

35. Subtract (of course algebraically) :—

(i) $2a + 3b - 4c$ from $5a - 3b + 2c$.
(ii) $5a - 3b + 2c$ from $2a + 3b - 4c$.
(iii) $a + b - c$ from $3(a - b + c)$.
(iv) $6{\cdot}2(p - q + 2)$ from $2{\cdot}3(2p - 2q + 1)$.
(v) $\frac{2}{3}\{2p - 3(q - 1)\}$ from $\frac{1}{3}\{4p - 3(r - 1)\}$.

EXERCISE XXIX.

DIRECTED PRODUCTS.

A.

1. York station is 32 miles to the north of Doncaster station on the Great Northern Railway. Write a formula for calculating the distance (d) of a train from Doncaster, given the time (t) since it left York. The train may be supposed to travel with a constant velocity v.

Use the formula to find the distance of the train from Doncaster in the following circumstances :—

(i) The train is travelling to the north at 35 mls./hr., and left York two and a half hours ago ;

(ii) The same, except that the train is travelling south ;

(iii) The train's velocity is 46 mls./hr. towards the north, but it will not reach York for one and a half hours ;

(iv) The train is coming from Scotland at 44 mls./hr., and will reach York in three-quarters of an hour ;

(v) The train is travelling southwards at 40 mls./hr., and left York forty-eight minutes ago.

2. A steamer, which trades between Cape Town and Melbourne, sails for many days along the 39th parallel (S.) at a uniform rate of v mls./hr. At a certain moment it is reported (by wireless telegraphy) to be at a given number of miles (d_0) from the island of St. Paul, which is on the same parallel. Write a formula for its position with regard to St. Paul (d) a given number of days (t) after or before this moment.

Find the position of the steamer :—

(i) Three days after it is reported to be 250 miles to the east of St. Paul on its westward journey and travelling at 12 mls./hr. ;

(ii) Fifteen hours after it is 120 miles to the west of St. Paul, pursuing a westward course at 15 mls./hr. ;

(iii) Ten hours before it is reported as in (ii).

3. The water in a reservoir has a depth of d_0 feet which is increasing h feet per day. Find the depth (d) after t days.

Use the formula to calculate the depth twelve days ago, given that the surface is sinking 3 inches daily and that the present depth is 13 ft.

4. A ship taking soundings in a bay finds that the depth of the sea at a certain point is d_0 fathoms and that it increases regularly by i fathoms for every quarter of a mile towards the west. Write a formula giving the depth, d, m miles west of the given point.

Use the formula to calculate the depth 4 miles to the east of a point where the depth is 400 fathoms, assuming that the soundings decrease in this direction at the rate of 2·4 fathoms per quarter of a mile.

5. A road climbs up a valley at a uniform slope of $a°$. A certain inn on the roadside is h_0 feet above sea-level. Write a formula giving the height (h) above sea-level at a point m miles along the road farther up the valley.

Use the formula to find the height above sea-level at a point (i) $2\frac{1}{2}$ miles before, (ii) 3 miles after the inn is reached, when you are walking downhill at a constant slope of 7°, and the inn is 2500 feet above the sea.

6. A farm-house, A, is situated on a road that runs east and west. Another house, B, lies p miles due north of A on a straight road that crosses the former at an angle of $a°$ some distance to the west of A. A third house, C, lies on the second road q miles from B in the opposite direction to the crossing. How far is C north of the road through A?

Given that the two roads cross at an angle of 37°, that B is $1\frac{1}{2}$ miles to the south of A, and that C is $2\frac{1}{4}$ miles from B in the direction of the crossing, find by the above formula the distance of C from the road through A.

Illustrate your solution by a diagram.

7. A charitable society has a balance of B_0 at the bank. Its revenue exceeds the expenses by an annual amount p. What will be the balance, B, after t years?

Suppose that the society has overdrawn its banking account to the amount of £130, but that it may be assumed that its regular income always exceeds the expenses by £20 per annum. Find by your formula the balance (i) 5 years hence, (ii) 6 years ago.

8. A motor-car is going with a velocity of u mls./hr., and this velocity is increasing regularly at such a rate that at the end of every minute the car goes a mls./hr. faster than at the

beginning of that minute. Give a formula for v, its velocity, after t minutes.

At the moment a motor-car passed me it was travelling at 27 mls./hr., but its speed had been diminishing and continued afterwards to diminish. Assuming that the rate of diminution of speed may be taken as 3 mls./hr. every minute, use the foregoing formula to calculate the speed of the car (i) in five minutes' time; (ii) four minutes ago; (iii) in nine minutes' time; (iv) in eleven minutes' time. What is the meaning of the last result?

9. Two motor-cars start in the same direction at the same moment and travel with uniform velocities v_1 and v_2. The first reaches a place X after t_1 hours, the second a place Y after t_2 hours. Give a formula for d, the distance between X and Y.

Two motor-cars pass one another on the road at exactly twelve o'clock. One is travelling at 24 mls./hr., the other at 20 mls./hr. in the opposite direction. The first car left X at 10.30, the second reached Y at 2.30. Assuming that the speeds were constant throughout, find the distance between X and Y by means of the foregoing formula.

10. A motor-car is travelling at u mls./hr., but its speed is increasing a mls./hr. every minute. In how many minutes will it be v mls./hr.?

(i) A motor-car is travelling at 28 mls./hr., but its speed is diminishing 4 mls./hr. every minute. Apply the foregoing formula to find how long it will be before its speed will be reduced to 8 mls./hr.

(ii) Assuming that the conditions remain constant, in how many minutes will it be travelling at 14 mls./hr. in the opposite direction?

B.

11. What laws of succession are exhibited by the terms of the following **sequences** of directed numbers?

(i) . . . $+4$, $+7$, $+10$, $+13$, $+16$, . . .
(ii) . . . $+18$, $+13$, $+8$, $+3$, . . .
(iii) . . . -8, -4, 0, $+4$, . . .

Continue each sequence three terms to the right and three terms to the left.

12. Write a formula for the term in No. 11 (i) which lies

n places to the right of $+10$. Use it to calculate the 50th term to the right of $+10$.

13. Write a formula for calculating the nth term to the left of $+10$ in the same sequence. What is the 60th term in this direction?

14. Show that by making n a directed number the formula of No. 12 can also be used to answer the question of No. 13. Use this formula to find (i) the 100th, (ii) the 200th term to the left of $+10$.

15. Write a single formula that can be used to calculate the nth term to the right or left of $+8$ in No. 11 (ii). Use it to find (i) the 20th term to the right, (ii) the 12th term to the left of $+8$.

16. Write a formula for the nth term of No. 11 (iii), counting forwards or backwards from -4. Calculate (i) the 1000th term to the left, (ii) the 1000th term to the right.

Note.—Sequences like those of No. 11 are called **arithmetic sequences**. Their characteristic is that each term is derived from the next term to the left of it by adding (algebraically) a constant number. This number is called the **common difference** of the sequence. What are the common differences of the sequences in No. 11?

17. The common difference of an arithmetic sequence is $+3$ and one of its terms is -7. What are the four terms respectively to the left and right (or before and after) this term?

Give the formula for the nth term of the sequence to the right or left of -7. Use it to find the 40th term to the left and the 15th to the right.

18. The common difference of an arithmetic sequence is -24 and one term is -72. Give the four terms immediately preceding and following this term. Calculate the 500th term to the left of -72.

Note.—The formula for the terms before and after (or to the left and right) of a given term may be called **the generating formula** of the sequence. The term from which the counting right and left proceeds may be called **the starting term**.

19. State the starting term and the common difference of the sequences whose generating formulæ are :—

(i) $T_n = -7\cdot2 + 3\cdot1n$. (ii) $T_n = +6\cdot7 - 1\cdot6n$.

(iii) $T_n = +2\frac{1}{2} - \frac{5}{8}n$. (iv) $T_n = -\frac{1}{8} - \frac{1}{20}n$.

Calculate in the case of (i) and (ii) the 100th term to the left of the starting term, and, in the case of (iii) and (iv), the 1000th term after the starting term.

20. Write the generating formulæ of the arithmetic sequences in which the common differences are respectively (i) $+ 18$, (ii) $- 8{\cdot}6$, (iii) $- 3\frac{3}{4}$, (iv) $+ 0{\cdot}07$, and the starting terms are respectively (i) $- 3$, (ii) 0, (iii) $+ 1\frac{1}{2}$, (iv) $- 0{\cdot}9$.

21. Give a generating formula that will apply to any arithmetic sequence. (Let $T_0 \equiv$ the starting term and $d \equiv$ the common difference.)

22. Counting to the right of the starting term the 20th term of an A.S. is $+ 17$ and the 30th term $+ 84$. Find the common difference, the starting term, and the generating formula.

23. In an arithmetic sequence $T_{12} = + 8$, $T_{20} = - 32$. Find the generating formula.

24. The 14th term to the right of the starting term of an A.S. is $+ 6{\cdot}7$, the 6th term to the left is $- 18{\cdot}3$. Find the generating formula.

25. The 17th term before the starting term of an A.S. is $+ 23{\cdot}6$; the 13th term after it is $- 57{\cdot}4$. What is the generating formula?

26. Find the generating formula of an A.S. in which $T_{-24} = + 19$ and $T_{-37} = + 58$.

27. In a certain A.S. $T_p = u$ and $T \ = v$. Express d, T_0, and the generating formula in terms of u, v, p, and q. Do your formulæ hold good for all possible values of p and q, u and v?

C.

Note.—Select any term from an unlimited arithmetical sequence. The series made up of this term and any number, limited or unlimited, of consecutive terms which immediately follow it is called an **arithmetical progression** (A.P.). The series composed of this term and any number of consecutive terms which immediately precede it is called an arithmetic **regression** (A.R.). Thus a progression always has a first term but may have no last term; while a regression always has a last term but may have no first term.

The numbers which constitute an A.P. are often said to be "in arithmetical progression". The terms which lie between

the first and last terms of an A.P. are called the **arithmetic means** between those terms. Thus, if an A.P. has 12 terms there are 10 arithmetic means.

28. There are 8 numbers in arithmetic progression, the first being 3 and the last 31. What is the common difference? What are the arithmetic means?

29. The first term of an A.P. is + 3, the last − 21, and there are 5 arithmetic means between them. What are they?

30. Given that the first and last terms of an A.P. containing n terms are respectively a and l, write a formula for d, the common difference.

31. There are n arithmetic means between two terms a and l. What is the common difference?

32. An A.P. contains n terms altogether. The first term is a and the common difference is d. Write expressions for the 2nd, 3rd, pth, and last (i.e. nth) terms. (Note the important difference between the first of a number of terms in A.P. and the "starting term" of an endless A.S. In counting the terms the former is included, the latter is not.)

33. The first of 21 numbers in A.P. is + 16·7 and the common difference is − 3·2. Find the last term, the first mean but one, and the last mean but two.

34. The third of 16 numbers in A.P. is 14, the thirteenth is 44. What are the first and last terms?

35. The common difference of an A.P. is − 3. The first and last terms are respectively + 23 and − 25. How many means are there?

36. Are any of the numbers + 5, 0, − 1, − 13, − 20, among the means in No. 35? If so, state their positions.

37. Is 72 a term of the A.P. 1, $3\frac{1}{4}$, $5\frac{1}{2}$, . . . If so, which term is it? If not, between which terms does it lie?

38. I make a series by taking the 2nd, 5th, 8th, etc., terms of an A.P. Will the new series be in A.P.? If so, what will be the common difference?

Show that the numbers obtained by taking every pth term of an A.P. are themselves in A.P.

39. Between each pair of terms of an A.P. p arithmetic means are inserted. Show that the whole series forms an A.P. If there were originally n terms, how many terms are there in the new series?

40. Taking a series of numbers in A.P. (i) I add the same number to each; (ii) I subtract the same number from

each; (iii) I multiply each by the same number; (iv) I divide each by the same number; (v) I multiply each by the same number and take another constant number away from the product; (vi) I square each; (vii) I take the square root of each. In which of these cases will the resulting numbers be in A.P.? Give full reasons. When the new series is in A.P., what will its common difference be?

EXERCISE XXX.

SUMMATION OF ARITHMETIC SERIES.

A.

1. Draw diagrams to illustrate the following summations :—

(i) 7 terms of the series $1 + 5 + 9 + \ldots$;
(ii) 6 terms of the series $24 + 21 + 18 + \ldots$;
(iii) 5 terms of the series $-3 + 1 + 5 + \ldots$;
(iv) 8 terms of the series $+13 + 8 + 3 - \ldots$.

2. Calculate the sum of :—

(i) The series $1 + 2 + 3 + \ldots + 1{,}000{,}000$;
(ii) The series $1 + 2 + 3 + \ldots + n$;
(iii) The series $1 + 3 + 5 + \ldots + 199$;
(iv) The first n odd numbers ;
(v) The first n even numbers.

3. Find the sum of the series :—

(i) $-8 - 1 + 6 + \ldots + 104$;
(ii) $92 + 80 + 68 + \ldots - 76$;
(iii) $21{\cdot}2 + 18{\cdot}3 + 15{\cdot}4 + \ldots - 33{\cdot}9$;
(iv) $\frac{1}{4} + \frac{7}{8} + 1\frac{1}{2} + \ldots + 14\frac{5}{8}$.

4. Sum the series :—

(i) $-1 - 11 - 21 - \ldots$ to 50 terms ;
(ii) $17 + 11 + 5 - \ldots$ to 21 terms ;
(iii) $-3{\cdot}4 - 1{\cdot}7 + 0 + \ldots$ to 8 terms ;
(iv) $-188{\cdot}8 - 185{\cdot}6 - 182{\cdot}4 - \ldots$ to 120 terms.

5. Find the sum of the A.S. in which the first and last terms and the number of terms are respectively— (i) 1, 101, 26 ; (ii) 1, 2, 101 ; (iii) -72, $+36$, 55 ; (iv) $+17{\cdot}3$, $-21{\cdot}7$, 51.

6. Write out the proof by symbols (i) that $S = \frac{n}{2}(a + l)$;

(ii) that $S = \frac{n}{2}\{2a + (n - 1)d\}$.

7. A man enters an office at a salary of £80, which is increased annually by £5. How much will the firm pay him in the course of twenty years?

8. In a "block race" the first block is 3 yards from the starting (and finishing) line; the others follow at regular distances of 5 feet. There are sixteen blocks altogether. What is the total distance that the blocks are carried?

9. The first week a restaurant is opened the proprietor's expenses exceeded his takings by £5 12s.; the second week the loss was £3 4s. If the improvement were maintained at the same rate, how much profit would the proprietor make altogether in 13 weeks?

B.

10. Three boys, A, B, and C, are apprenticed, each for six years, under the following conditions. A is to receive nothing during the first year, £10 the second year, £20 the third year, etc., and £50 the last year. B is to receive nothing during the first six months. His wages are then to rise by equal half-yearly steps so that during the last six months he will be paid like A, at the rate of £50 a year. C is to serve for nothing during the first three months. His wages will then be increased uniformly every three months by such an amount that during the last quarter of the sixth year he also will be paid at the rate of £50 a year.

Draw (to the same scale) "column graphs" exhibiting the wages received by A, B, and C respectively at different periods. Write out a proof by means of the graphs that during the five years each apprentice will receive the same total amount in wages. What is this amount? What were B's half-yearly and C's quarterly increments?

11. For a minute after a tap delivering into a tank is turned on no water flows. It then pours out at the rate of 2 gals./min. At the end of a minute the tap is suddenly turned on farther so that it now delivers 4 gals./min. After another minute the tap is again turned and the flow increases to 6 gals./min. After one more minute the flow is increased to 8 gals./min. At the end of the fifth minute the tap is suddenly turned off completely. Draw a rectangle graph exhibiting the quantity of water supplied in each minute. (Make each rectangle 1 inch wide.) What is the total quantity of water delivered?

12. On another occasion the tap was turned on farther every six seconds for five minutes, in such a way that the

flow increased by equal steps. The rate of delivery during the last six seconds was 8 gals./min. During the first six seconds no water was delivered. Erect upon the base line of the former graph a column graph exhibiting the new conditions of flow. What is the total quantity of water delivered?

13. On a third occasion the tap was turned on gradually and without a break, so that the flow increased constantly and uniformly. At the end of five minutes when the tap was suddenly turned off the rate of delivery had reached 8 gals./min. Using the same base line as in Nos. 12 and 13, draw a graph exhibiting the conditions of flow. How much water was delivered in five minutes? How much did the rate of flow increase in the course of each minute?

14. In Nos. 10, 11, 12, 13 what line measures the rate of wages or the rate of flow at a given moment?

15. Show that in No. 13 the rate of flow at any moment is given by the formula

$$r = 1{\cdot}6\ t$$

r being the rate of flow in gallons per minute and t the time in minutes (and fractions of a minute) since the tap was turned on. What was the rate of flow after 4·2 minutes?

16. Show that in No. 13 the number of gallons (Q) delivered in t minutes is given by the formula

$$Q = 0{\cdot}8\ t^2$$

How much was in the tank at the end of 3·3 minutes?

17. For a considerable time one morning the state of the crowd passing over London Bridge was given by the formula

$$r = 180\ t$$

r being the rate of movement of the crowd measured by the number of persons per *hour* who would have passed a certain point if the rate had been maintained; t being the time in *minutes* since the observations began. Show that the total number who passed a certain point in the first t minutes is given by the formula

$$N = \tfrac{3}{2}\ t^2$$

18. The speed of a marble permitted to roll from rest down a smooth slope increases uniformly. After ten seconds it is 3 feet per second. Draw a graph showing the speed at different times. What does a unit of area in this graph represent? Find from the graph the distance the marble rolls (i) in five seconds, (ii) in ten seconds. Write formulæ

for the speed acquired (v) and the distance rolled (s) in t seconds.

19. A railway train starts from rest and increases its speed uniformly until after twelve minutes it is going at 60 miles an hour. Draw a graph showing the speed and the distance travelled in a given time. What distance is represented in this graph by the unit of area? Write formulæ for v, the speed in miles per hour after t minutes, and for s the distance in miles travelled in t minutes. What is the speed after eight minutes, and how far has the train then travelled?

20. The speed of a moving body in feet per second is given by the formula

$$v = at$$

t being the time in seconds since it began to move. What formula gives the distance in feet travelled in t seconds? What does this formula become if t is measured in seconds but v in feet per minute?

21. A motor-car is moving at the rate of 15 mls./hr. when it reaches a descent down which its speed increases regularly at the rate of 3 mls./hr. per minute. Draw a graph showing its speed at any time, measured from the moment when it began to increase. Write formulæ for the values of (i) v (measured in miles/hour), (ii) v (measured in miles/minutes), and (iii) s (measured in miles) after a given time t (measured in minutes).

22. The speed of a moving body is given by the formula

$$v = u + at$$

u and v being measured in feet per second and t in seconds. Write a formula for the distance moved in t seconds. What does the formula become if u and v are measured in miles per hour and t in minutes?

23. Show by a diagram that if $v = u - at$, then $s = ut - \frac{1}{2}at^2$, the same unit of time being used in measuring u, v, and t.

24. A train is travelling at 40 mls./hr. when the brakes are put on. The speed now decreases uniformly and in five minutes the train is at rest. Draw a graph exhibiting its speed and the distance travelled from the moment when the brakes are put on. Give formulæ for v and s, t being measured in minutes.

25. A boy rolls a ball up a smooth slope with a speed of 18 ft./sec. It gets uniformly slower, stops, and then rolls

down again with uniformly increasing speed. The decrease and increase of the speed are both at the rate of 3 ft./sec. every second. Draw a diagram showing the speed of the ball and the distance travelled at any time for forty seconds after it is thrown. [How will you distinguish between upward and downward speed? Between distance up and distance down the slope?]

26. From the graph answer the following questions :—

(i) When will the ball have exhausted its speed?
(ii) How far will it then have travelled?
(iii) What will be its speed twelve seconds after it is thrown?
(iv) Where will it then be?
(v) What will be its speed twenty seconds after it is thrown?
(vi) Where will it then be?

27. Show that you can obtain the answers to No. 26 from the formulæ $v = u + at$ and $s = ut + \frac{1}{2}at^2$, by substituting directed numbers for the symbols.

28. A bullet is fired up into the air with a speed of 1600 ft./sec. Its speed falls off as it goes up and increases as it comes down at a regular rate of 32 ft./sec. every second. Calculate (i) how long it will be rising; (ii) how high it will rise; (iii) its velocity after twenty seconds; (iv) its height above the ground at that moment; (v) its velocity after sixty seconds; (vi) its position then.

29. The following formulæ give the velocities of three bodies, A, B, and C, which may be supposed to be moving along lines parallel to the lines across your paper. Write down the corresponding formulæ for their distances at a given moment from the point from which the measurements are taken. In each case v is measured in centimetres per second, and t in seconds.

(i) $v = 7{\cdot}2 - + 6{\cdot}4t$; (ii) $v = -10 + 8{\cdot}6t$; (iii) $v = -8{\cdot}5 - 3{\cdot}6t$.

30. Answer by means of your formulæ the following questions with regard to the bodies A, B, and C :—

(i) Which way and with what speed was each body moving ten seconds ago?
(ii) Where were they ten seconds ago?
(iii) How was each moving one second ago?
(iv) Where was each body one second ago?
(v) Each body has been or will be motionless for an instant. Find when this happened or will happen in each case. What did or will each body do immediately after these moments?

Fig. 33.

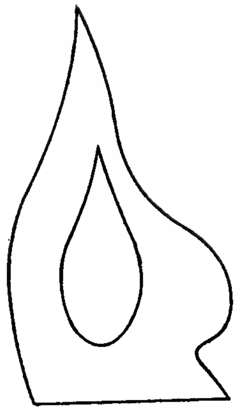


Fig. 34.

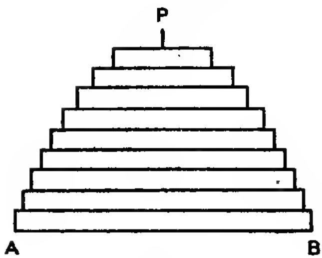


Fig. 35.

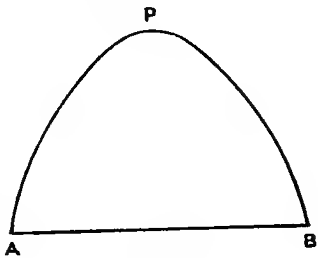


Fig. 37.

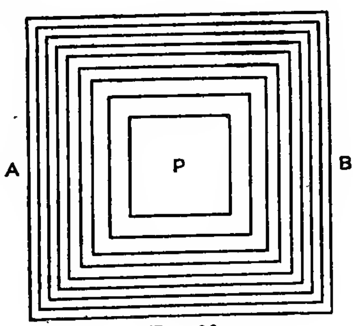


Fig. 36.

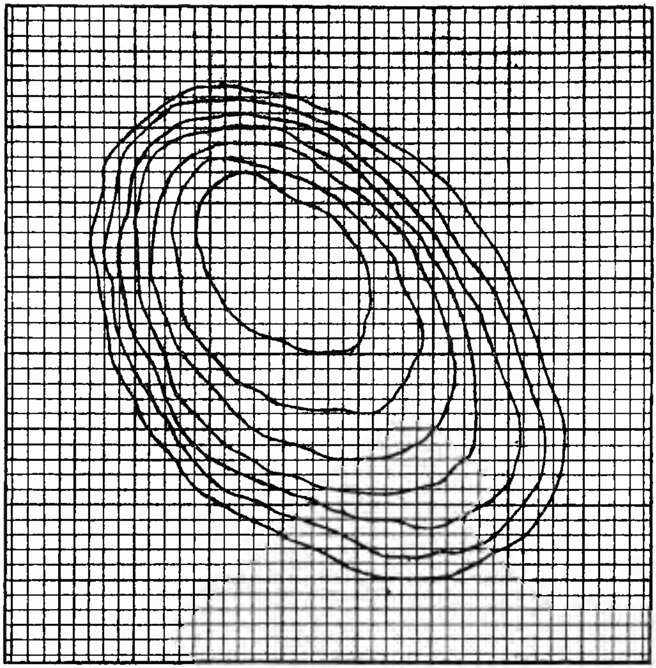


FIG. 38.

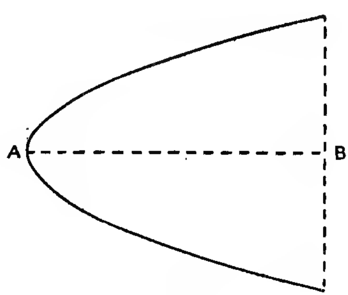


FIG. 39.

C.

31. Determine by measurements taken at random whether equidistant lines drawn across fig. 33 are in A.P. If they are calculate the area of the figure in sq. cms. [See Ex. XXIX., No. 38.]

32. Do the same with fig. 34.

33. A number of square slabs 2 inches thick are piled on top of one another so as to produce the solid figure shown in elevation in fig. 35. A pin 2 inches high, but of negligible volume, surmounts the pile. The squares in fig. 36 are the outlines of the successive blocks as seen from above. Calculate the differences between the areas of successive squares, counting the pin as having zero area. Hence calculate the volume of the solid.

34. Fig. 37 shows an elevation of a pointed solid standing on a square base. Fig. 36 may be regarded as a "contour map" of this solid, the squares being contour lines taken at intervals of 2 inches measured vertically. What is the probable volume of the solid? Upon what assumption is your calculation based?

35. Fig. 38 is a contour plan of a heap of gravel. Each small square in the plan represents an area of 1 square foot. The contours are taken at distances of 1 foot measured vertically. The height of the heap is 7 feet. Calculate its probable volume.

36. Fig. 39 is the side outline of the reflector of a searchlight drawn one-fourth of the actual linear size. Its cross-section is everywhere circular. On AB choose, at random, any number of equidistant points. Measure the radii of the circular cross-sections at the points so determined. Find by calculation whether their areas are in A.P. If they are, calculate the volume enclosed by the reflector.

EXERCISE XXXI.

ALGEBRAIC MULTIPLICATION.

A.

1. Test by diagrams the identity $(a - b)\,c$, a and b being negative, c positive, and a numerically greater than b.

2. Test by diagrams the identity $(a + b)^2 = a^2 + 2ab + b^2$ when a is negative, b positive, and a numerically greater than b.

3. Draw a set of diagrams to illustrate and test the steps by which the identity can be proved by multiplication, a and b being limited as in No. 2. [Study the frontispiece.]

4. Test by diagrams the identity $(a - b)^2 = a^2 - 2ab + b^2$ when a and b are both negative and a is numerically less than b.

5. Draw diagrams to illustrate and test the proof of the identity by multiplication, a and b being limited as in No. 4.

B.

6. Complete the following identities:—

(i) $(2a + 3b)(3a - 2b) =$; (ii) $(4p - 3q)(2p - 3q) =$;
(iii) $(2x - 5y)(3x - 4y) =$; (iv) $(ax + by)(bx + ay) =$;
(v) $(ax - by)(bx - ay) =$; (vi) $(ax - by)(bx + ay) =$;
(vii) $\left(\frac{3}{4}x - 1\right)\left(\frac{1}{4}x - 1\right) =$; (viii) $\left(\frac{a}{b}x - p\right)\left(\frac{b}{a}x - q\right) =$;
(ix) $\left(\frac{2}{a} + \frac{3}{b}\right)\left(\frac{3}{a} - \frac{2}{b}\right) =$; (x) $\left(\frac{a}{x} + \frac{b}{y}\right)\left(\frac{b}{x} - \frac{a}{y}\right) =$.

7. Test your answer to No. 6 (i) by putting $a = -3$, $b = +4$.

8. Test your answer to No. 6 (v) by putting $x = +2$, $y = -4$, $a = -3$, $b = +5$.

9. Test your answer to No. 6 (viii) by putting $a = +2$, $b = -3$, $p = -8$, $q = +7$, $x = -10$.

10. Show that for a given value of n, positive or negative, the expressions $2n + 1$ and $2n - 1$ always describe con-

secutive odd numbers. What expression will describe the product of two consecutive odd numbers?

Test the validity of the expression when $n = 10$ and when $n = -4$.

Prove that the product is itself always an odd number.

11. Prove that the product of any two odd numbers, positive or negative, is always odd. (Take $2p + 1$ and $2q + 1$ as expressions for the two odd numbers.)

12. Two numbers, N_1 and N_2, have respectively the forms $n^2 - n + 1$ and $n + 1$. What is the form of their product?

Make a table of the values of N_1 and N_2, when $n = +3$, $+2$, $+1$, 0, -1, -2, -3 respectively. Do the products follow the calculated law?

13. Two numbers are respectively of the forms $n^2 + n + 1$ and $n - 1$. What is the form of their product? Test the result by putting $n = -2{\cdot}3$.

14. What is the form of the product of two numbers which have respectively the forms $n^2 + n + 1$ and $n^2 - n + 1$? Confirm in the cases when $n = 0$, $+3$, and -10.

15. Show that $\frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = a^2 + b^2 + c^2 - ab - bc - ca$. Test by putting $a = -2$, $b = +3$, $c = -1$. Select any three values you please for a, b, and c and test again.

16. Why must a number of the form $a^2 + b^2 + c^2 - ab - bc - ca$ always be positive? Test the statement by substitution.

17. Two numbers, N_1 and N_2 have respectively the forms $a^2 + b^2 + c^2 - ab - bc - ca$ and $a + b + c$. Find the form of their product. Prove that the sign of the product is the same as the sign of N_2.

18. Prove that $ab(a + b) + bc(b + c) + ca(c + a) = -\frac{1}{3}\{(a - b)^3 + (b - c)^3 + (c - b)^3\}$. Test when $a = +1$, $b = -1$, $c = -2$.

C.

Note.—The meaning of a number of several digits, such as 2437, can be expressed as follows:—

$$\begin{aligned} 2437 &= 2 \times 1000 + 4 \times 100 + 3 \times 10 + 7 \\ &= 2 \times 10^3 + 4 \times 10^2 + 3 \times 10 + 7 \\ &= 2t^3 + 4t^2 + 3t + 7 \end{aligned}$$

the symbol t being substituted for 10 for the sake of conciseness.

19. Write in ordinary notation the following numbers :—

(i) $3t^3 + 7t^2 + 5t + 6$. (ii) $2t^4 + 9t^3 + 6t^2 + 3t + 1$.
(iii) $3t^5 + 5t^3 + 2t^2 + 6$. (iv) $t^6 + 2t^4 + 2t^2 + 1$.
(v) $t^3 - 3t^2 + 2t - 5$. (vi) $t^5 - t^4 + t^3 - t^2 + t - 1$.

20. Find the form of the product of two numbers whose forms are $t^2 + 2t + 3$ and $3t + 2$ respectively. Compare the process with that of multiplying 123 by 32.

21. Show that 123 may be expressed as $2t^2 - 8t + 3$, and 32 as $4t - 8$. Find the form of the product of the numbers expressed thus, and find whether it expresses the arithmetical result correctly.

22. Find the forms of the products of the pairs of numbers whose forms are given by the following expressions :—

(i) $2t^5 - 3t^4 + 4t^3 - 3t^2 + 2t - 1$ and $3t - 2$.
(ii) $3t^2 - 2t + 1$ and $t^2 - 2t + 3$.
(iii) $t^3 + 2t^2 + 2t + 1$ and $t^2 - 2t + 1$.
(iv) $16t^4 - 8t^3 + 4t^2 - 2t + 1$ and $2t + 1$.
(v) $t^4 + 3t^3 + 9t^2 + 27t + 81$ and $t - 3$.
(vi) $2t^3 - 3t^2 + 1$ and $3t^3 - 2t - 3$.

Confirm (iv) and (v) by arithmetic.

23. Obtain the algebraic product of the following expressions :—

(i) $a^4 - a^2b^2 + b^4$ and $a^2 + b^2$.
(ii) $a^3 + a^2b^2 + b^3$ and $a^3 - a^2b^2 + b^3$.
(iii) $x^2 - xy + x + y^2 + y + 1$ and $x + y - 1$.
(iv) $\frac{1}{a^3} - \frac{1}{a^2} + \frac{1}{a} - 1$ and $\frac{1}{a} + 1$.
(v) $\frac{4p^2}{9q^4} + \frac{2p}{3q^2} + 1$ and $\frac{4p^2}{9q^4} - \frac{2p}{3q^2} + 1$.

24. A number has the form $a + b + c$. What is the form of its square? Test in the case when $a = +3$, $b = -2$, $c = -3$.

25. Use the result of No. 24 to write down without multiplication the squares of the following expressions :—

(i) $a - b + c$. (ii) $a + b - c$.
(iii) $2a + 3b + c$. (iv) $2a + 3b - c$.
(v) $p - 2q + 3$. (vi) $3p - 2q + 1$.
(vii) $\frac{a}{b} + 2 + \frac{b}{a}$. (viii) $a^2 + b^2 + c^2$.
(ix) $p^3 + q^3 + r^3$. (x) $\frac{1}{p} - \frac{2}{q} + \frac{3}{r}$.

D.

26. Complete Stifel's Table as far as the row giving the coefficients of $(a + b)^{10}$.

27. Write down the expansion of $(a + b)^7$. Deduce from it the expansion of $(a - b)^7$.

28. Write down the expansion of $\left(1 + \frac{a}{2}\right)^6$ and of $\left(2p - \frac{q}{r}\right)^5$.

29. Demonstrate the following properties of the binomial coefficients :—

(i) The sum of the coefficients of $(a + b)^n$ is always 2^n. (Put $a = b = 1$) ;

(ii) The sum of the odd coefficients is always equal to the sum of the even coefficients ;

(iii) The coefficients succeed in the same order counting either from the first or from the last ;

(iv) If n is even there is one greatest coefficient, if odd, there are two equal greatest coefficients.

30. Calculate :—

(i) The fourth term in the expansion of $(3a - 1)^8$;

(ii) The third term in the expansion of $\left(1 - \frac{2}{3}p^2\right)^{10}$;

(iii) The term involving a^6 in the expansion of $(a - 3b)^9$;

(iv) The middle term in the expansion of $(p^2 - qr)^6$;

(v) The term containing no variable in the expansion of

$$\left(\frac{a}{2} + \frac{3}{a}\right)^8 ;$$

(vi) The terms containing p and $\frac{1}{p}$ in the expansion of

$$\left(ap - \frac{2b}{p}\right)^7 .$$

EXERCISE XXXII.

THE INDEX NOTATION.

Note.—Approximate answers should (in the absence of other instructions) be given in the standard form and should be correct to the first decimal place of the unit.

A.

1. The acreage and population in 1911 of the four largest and four smallest counties of England are given in the following table. Rewrite the table in the index notation, substituting approximate numbers and taking one million acres and one hundred thousand people as the unit. Two decimal places of the unit should be retained.

County.	Acres.	Population.
York . . .	3,721,339	3,969,151
Lincoln . .	1,668,603	557,543
Devon . . .	1,663,467	701,981
Northumberland .	1,291,515	697,014
Bedford . .	307,338	197,660
Huntingdon . .	207,572	48,105
Rutland . .	108,700	21,168
London . .	74,816	4,522,961

2. From your table calculate approximately the total acreage and population (i) of the four largest, (ii) of the four smallest counties. Calculate approximately the average acreage and population of each of the two groups of counties.

3. Express the following numbers in the ordinary notation: $41{\cdot}082 \times 10^5$, $7{\cdot}201 \times 10^2$, $0{\cdot}002871 \times 10^8$, $4{\cdot}20576 \times 10^3$, $0{\cdot}00034792 \times 10^4$.

4. Express the following numbers in the standard form: 23,827; 26·28; 41,200,000; 928·4; $147{\cdot}6 \times 10^7$; 1659×10^5; $0{\cdot}976 \times 10^7$; $0{\cdot}00325 \times 10^9$.

B.

5. Light travels at the rate of about 186,000 miles per second and takes about two and a half years to reach the nearest fixed star. Calculate the approximate distance of that star.

6. The Atlantic Ocean covers about 31,530,000 square miles and has an average depth of about 12,000 feet. A cubic foot of sea water weighs 64·2 lb. Find the approximate weight of the ocean in pounds.

7. The average population of the British Isles during the last ten years may be taken to be 43,600,000. Each person on the average breathes about 16 times a minute and at each breath draws in about 25 cubic inches of air. Find roughly how many cubic inches of air have passed in and out of British lungs during the last ten years.

8. Express each of the factors of the following products in the standard form and obtain the value of the products to three significant figures. Express the products also in the standard form :—

(i) $18{,}360 \times 5{,}018$.
(ii) $31{,}069 \times 283 \times 8{,}204{,}921$.
(iii) $3{\cdot}2 \times 1{,}921 \times 723{,}431 \times 17{\cdot}9$.
(iv) $136 \times 10^5 \times 0{\cdot}0372 \times 10^6$.
(v) $18{\cdot}03 \times 10^3 \times 0{\cdot}741 \times 10^2 \times 0{\cdot}000503 \times 10^8$.

C.

9. The distance of the earth from the sun is about 93,000,000 miles. How many minutes does the sun's light take to reach us? (See No. 5.)

10. Calculate roughly, from the result of No. 6, the number of tons in the Atlantic Ocean.

11. Calculate roughly, from the result of No. 7, the number of cubic miles of air breathed by the inhabitants of the British Isles during the last ten years.

12. Throw the numbers in the following expressions into the standard form and find the value of each expression to three significant figures, giving it in the standard form :—

(i) $\dfrac{18324 \times 927 \times 3491}{17024 \times 8189 \times 21}$.

(ii) $\dfrac{182 \times 10^4 \times 9{\cdot}6 \times 10^2 \times 0{\cdot}23 \times 10^3}{1209 \times 0{\cdot}048 \times 10^5}$.

(iii) $\dfrac{11{\cdot}3 \times 10^8 \times 0{\cdot}008 \times 10^5}{40{\cdot}03 \times 10^3 \times 565 \times 0{\cdot}02 \times 10^4}$.

D.

13. Simplify the following expressions :—

(i) $a^5b^3 \times bc^4 \div a^3bc^3$.

(ii) $\dfrac{a^2p^3}{b^3q^4} \times \dfrac{aq^2}{bp^3} \div \dfrac{p^3q^3}{a^2b^2}$.

(iii) $\left(\dfrac{a^2bpq}{b^2cr^2}\right)^2 \cdot \left(\dfrac{b^2cqr}{c^2ap^2}\right)^2 \cdot \left(\dfrac{c^2arp}{a^2bq^2}\right)^2$.

(iv) $\left(\dfrac{2pq}{3r}\right)^3 \cdot \left(\dfrac{3qr}{4p}\right)^3 \cdot \left(\dfrac{4rp}{5q}\right)^3$.

(v) $(a^3)^2$.

(vi) $(a^2)^3$.

(vii) $[\{(p^2)^4\}^3]^5$.

(viii) $[\{(p^5)^4\}^2]^3$.

14. Simplify the following expressions :—

(i) $\sqrt{a^2}$. (ii) $\sqrt{a^8}$.

(iii) $\sqrt{(a^4/b^2)}$. (iv) $\sqrt{(p^2x^2/q^6)}$.

(v) $\sqrt{\left\{\dfrac{a^3p^2q}{cr^3} \times \dfrac{b^3q^2r}{ap^3} \times \dfrac{c^3r^2p}{bq^3}\right\}}$. (vi) $\sqrt[3]{a^3}$.

(vii) $\sqrt[3]{p^6}$. (viii) $\sqrt[3]{\{(p^3r)^2/(q^3r^2)^7\}}$.

(ix) $\sqrt{\{(a^3b^4)^2 \times (ab^4)^3 \times a^3\}}$.

15. Simplify the following expressions :—

(i) $(x^2)^l \times (y^2)^m \div (z^2)^n$. (ii) $(xy)^l \cdot (yz)^m \cdot (zx)^n$.

(iii) $(a^2b)^l \cdot (b^2c)^m \cdot (c^2a)^n$. (iv) $\left(\dfrac{a^m}{b^n}\right)^l \left(\dfrac{b^n}{c^l}\right)^m \left(\dfrac{c^l}{a^m}\right)^n$.

(v) $\sqrt[n]{(a^n)^l}$.

16. Find the value of the following expressions :—

(i) $\dfrac{(182)^4 \times (14{\cdot}2)^3 \times 3000}{(9{\cdot}1)^4 \times (7{\cdot}1)^3}$.

(ii) $\dfrac{(5{\cdot}6)^3 \times (0{\cdot}024)^2 \times (0{\cdot}7)^3}{(0{\cdot}28)^4 \times (4{\cdot}2)}$.

(iii) $\dfrac{(0{\cdot}9)^3 \times (4{\cdot}9)^4 \times (2{\cdot}7)^2}{(6{\cdot}3)^5 \times 4{\cdot}5 \times 10^3}$.

17. Express each number in the following expressions as the product of powers of its prime factors. Reduce each expression to its simplest form as a product or ratio of such powers :—

(i) $1890 \times 24500 \times 504$.

(ii) $\dfrac{18 \times 1183 \times 520 \times 8670}{255 \times 663 \times 7650}$.

(iii) $\dfrac{2299 \times 253 \times 396}{247 \times 1472 \times 11362}$.

18. Write out in full a justification of each of the following equivalences. (The symbols m and n represent integers without sign) :—

(i) $\times\ a^m \times a^n = \times\ a^{m+n}$.

(ii) $\times\ a^m \div a^n = \times\ a^{m-n}$ if $m > n$.

$\times\ a^m \div a^n = \div\ a^{n-m}$ if $m < n$.

(iii) $\times\ (a^m)^n = \times\ a^{mn} = \times\ (a^n)^m$.

(iv) $\times\ \sqrt[n]{a^p} = \times\ a^{p/n}$ if p is an exact multiple of n.

EXERCISE XXXIII.

NEGATIVE INDICES.

A.

1. Express the following numbers in the standard form by means of negative powers of ten: 0·0248, 0·000372, 0·00006781, $\frac{5}{8}$, $\frac{3}{160}$, $\frac{1}{8000}$, $36/10^5$, $0{\cdot}23/10^4$, $1/(4 \times 10^7)$, $1/(250 \times 10^6)$.

Note.—The answers to Nos. 2-5 are to be given in the standard form and correctly to three significant figures.

2. From the table in Ex. XXXII, No. 1, calculate approximately the number of persons per acre in each of the four smallest counties.

3. In 1850 the French experimenter Foucault measured the speed of light by finding how long it took to travel 20 metres. Assuming the speed to be 300,000 kilometres/sec., what was the interval of time which Foucault measured?

4. The weight of the column of air resting on a square inch of the surface of the sea is about 15 lb. The air is supposed to reach to a height of about 200 miles. Calculate approximately the average weight of a cubic inch of the air above the sea.

5. A thousand feet above the sea the weight of the column of air resting on a square inch is about $14\frac{1}{2}$ lb. Find the average weight per cubic inch of the lowest thousand feet of air above the sea.

B.

6. Find, in the standard form, the value of each of the following expressions:—

(i) $\dfrac{7{\cdot}2 \times 10^3}{180 \times 10^5}$.

(ii) $8{\cdot}7 \times 10^4 \times 2{\cdot}5 \times 10^{-6} \times 800$.

(iii) $\dfrac{1{\cdot}4 \times 10^{-3} \times 2{\cdot}7 \times 10^2}{4{\cdot}2 \times 10^{-7} \times 8{\cdot}1 \times 10^3}$.

(iv) $\dfrac{3{\cdot}4 \times 10^4 \times 5{\cdot}7 \times 10^{-6} \times 4{\cdot}4 \times 10^{-3}}{1{\cdot}87 \times 10^{-4} \times 9{\cdot}5 \times 10^3 \times 3 \times 10^{-4}}$.

7. Express each of the following as a product of powers of prime numbers :—

$$\text{(i)}\quad \frac{9 \times 63 \times 51}{81 \times 14 \times 867}.$$

$$\text{(ii)}\quad \frac{23 \times 32 \times 65}{338 \times 598 \times 256 \times 115}.$$

$$\text{(iii)}\quad \frac{88 \times 143 \times 5071}{2299 \times 922 \times 64}.$$

8. Rewrite the following expressions with positive indices :—

(i) $a^3b^{-4}c^2$.

(ii) $4a^{-5}b^3c^{-4} \div 6a^2b^{-2}c^{-4}$.

(iii) $15p^2q^{-3} \div 9q^2r^{-3} \times 12r^2p^{-3}$.

(iv) $a^{-2}(bc^{-1} - b^{-1}c) + b^{-2}(ca^{-1} - c^{-1}a) + c^{-2}(ab^{-1} - a^{-1}b)$.

Give the last answer in a simplified form.

9. Express each of the following expressions in the form $Aa^mb^nc^p$. . . . (All symbols used as indices represent whole numbers) :—

(i) $\dfrac{4p^2q^3}{5q^2r^5}$. (ii) $1/\sqrt{x^6}$.

(iii) $6a^{-4}b^7 \div 3\sqrt{a^{-2}b^8}$. (iv) $\left(\dfrac{x^2}{y^3}\right)^l . \left(\dfrac{y^2}{z^3}\right)^m . \left(\dfrac{z^2}{x^3}\right)^n$.

(v) $\left(\dfrac{x^m}{y^n}\right)^l . \left(\dfrac{y^n}{z^l}\right)^m . \left(\dfrac{z^l}{x^m}\right)^n$.

10. Write out a justification of the equivalences :—

(i) $a^{-n} = \dfrac{1}{a^n}$, (ii) $a^0 = 1$,

n being a positive whole number.

EXERCISE XXXIV.

FACTORIZATION.

A.

1. Each of the following expressions gives an algebraic product and one of its factors. Find the other factor.

(i) $(25p^2 - 121q^2)/(5p - 11q)$.
(ii) $(a^2 + 6ab + 9b^2)/(a + 3b)$.
(iii) $(16m^2 - 24mn + 9n^2)/(4m - 3n)$.
(iv) $\left(a^2 - \frac{2a}{b} + \frac{1}{b^2}\right)\Big/\left(a - \frac{1}{b}\right)$.
(v) $\left(4a^2 + \frac{12a}{b} + \frac{9}{b^2}\right)\Big/\left(2a + \frac{3}{b}\right)$.
(vi) $(p^2 + 7p + 12)/(p + 3)$.
(vii) $(p^4 - 25p^2 + 156)/(p^2 - 12)$.
(viii) $(a^2x^4 - 2ax^2 - 80)/(ax^2 + 8)$.
(ix) $(x^4 + 2ax^2 - 80a^2)/(x^2 + 10a)$.
(x) $\left(\frac{1}{9}.\frac{p^2}{q^2} - \frac{5}{3}.\frac{p}{q} - 126\right)\Big/\left(\frac{1}{3}\,\frac{p}{q} - 14\right)$.
(xi) $(a^3 - 3a^2 + 3a - 1)/(a - 1)$.
(xii) $\left(a^3 + \frac{3}{2}a^2b + \frac{3}{4}.ab^2 + \frac{b^3}{8}\right)\Big/\left(a + \frac{b}{2}\right)$.
(xiii) $\left(\frac{8}{p^3} - \frac{4q}{p^2} + \frac{2}{3}.\frac{q^2}{p} - \frac{q^3}{27}\right)\Big/\left(\frac{2}{p} - \frac{q}{3}\right)$.
(xiv) $(8a^3 + 27)/(2a + 3)$.
(xv) $\left(\frac{p^3}{64} + 1\right)\Big/\left(\frac{p}{4} + 1\right)$.
(xvi) $(a^3p^6 - q^6)/(ap^2 - q^2)$.
(xvii) $\left(1 - \frac{125}{27r^3}\right)\Big/\left(1 - \frac{5}{3r}\right)$.
(xviii) $(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca)/(a + b + c)$.
(xix) $(p^2 + q^2 + r^2 - 2pq - 2qr + 2rp)/(p - q + r)$.
(xx) $(a^3 + b^3 + c^3 - 3abc)/(a + b + c)$.
(xxi) $(a^3 + b^3 - c^3 + 3abc)/(a + b - c)$.
(xxii) $(x^3 - y^3 - z^3 - 3xyz)/(x - y - z)$.

2. Factorize the following expressions :—

(i) $a^2 + 9a + 20$.
(ii) $x^2 - 9x + 20$.
(iii) $a^2 - 15ab + 56b^2$.

(iv) $p^2x^4 + 23px^2 + 130$.
(v) $12x^4 - 7px^2 + p^2$.
(vi) $6a^2 + 13ab + 6b^2$.
(vii) $6a^2 - 17ab + 12b^2$.
(viii) $8x^2 - 38x + 35$.
(ix) $4x^2 + 28x + 49$.
(x) $16x^4 - \frac{40x^2}{y} + \frac{25}{y^2}$.
(xi) $a^2 + 2a - 8$.
(xii) $a^2 - 2a - 8$.
(xiii) $p^2 + 3pq - 28q^2$.
(xiv) $a^2p^2 - 3ap - 28$.
(xv) $20a^2 + a - 1$.
(xvi) $20p^2 - pq - q^2$.
(xvii) $6x^2 + 5x - 6$.
(xviii) $6a^2 - 5ab - 6b^2$.
(xix) $21\frac{a^2}{p^4} + 23\frac{a}{p^2} - 20$.
(xx) $12 - 4p^2 - 21p^4$.
(xxi) $39 - 46\frac{a}{x^2} - 8\frac{a^2}{x^4}$.
(xxii) $8a^3 - 27$.
(xxiii) $1 + \frac{x^3}{64}$.
(xxiv) $27a^3 + 64b^3$.
(xxv) $p^3 - \frac{8}{p^3}$.
(xxvi) $\frac{1}{27a^6} + \frac{125}{b^6}$.
(xxvii) $x^4 + x^2y^2 + y^4$.
(xxviii) $\frac{a^4}{16} + \frac{9a^2}{4} + 81$.
(xxix) $n^8 + n^4 + 1$.
(xxx) $8p^6 + 63p^3 - 8$.

B.

3. Write down by inspection the algebraic square root of each of the following expressions :—

(i) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
(ii) $a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$.
(iii) $p^2 + 9q^2 + 4r^2 + 6pq + 12qr + 4rp$.
(iv) $4a^2 + 9b^2 + 25c^2 + 12ab - 30bc - 20ca$.
(v) $\frac{a^2}{4} + \frac{b^2}{9} + \frac{c^2}{25} - \frac{1}{3}ab - \frac{2}{15}bc + \frac{1}{5}ca$.
(vi) $\frac{1}{4}p^4 + \frac{4}{9}q^4 + \frac{9}{16}r^4 + \frac{2}{3}p^2q^2 + q^2r^2 + \frac{3}{4}r^2p^2$.
(vii) $a^2b^2 + 4b^2c^2 + 16c^2a^2 - 4ab^2c - 16bc^2a + 8ca^2b$.

Note.—If we expand $(a^2 + a + 2)^2$ by the formula we have :—

$$(a^2 + a + 2)^2 = a^4 + a^2 + 4 + 2a^3 + 4a + 4a^2$$
$$= a^4 + 2a^3 + 5a^2 + 4a + 4$$

In the second line (i) the two terms containing a^2 have been taken together and (ii) the terms have been rearranged in descending powers of a. If, then, an expression like that in the second line is given and its square root is required, the terms must be rearranged and decomposed so as to fall into the scheme of the first line.

4. Find the algebraic square root of the following expressions :—

(i) $a^4 - 2a^3 + 5a^2 - 4a + 4.$

(ii) $a^4 + 2a^3 - 3a^2 - 4a + 4.$

(iii) $a^4 - 4a^3 + 10a^2 - 12a + 9.$

(iv) $4a^4 + 20a^3 + \frac{71}{3}a^2 - \frac{10}{3}a + \frac{1}{9}.$

(v) $\frac{a^4}{9} - \frac{2}{3}a^3 + 3a^2 - 6a + 9.$

Note.—$$\left(a + 2 + \frac{1}{a}\right)^2 = a^2 + 4 + \frac{1}{a^2} + 4a + \frac{4}{a} + 2$$
$$= a^2 + 4a + 6 + \frac{4}{a} + \frac{1}{a^2}.$$

The second line could be written

$$a^2 + 4a^1 + 6a^0 + 4a^{-1} + a^{-2}.$$

This way of writing it shows that the terms are arranged in descending powers of a.

5. Find the algebraic square root of the following expressions :—

(i) $p^2 - 4p + 6 - \frac{4}{p} + \frac{1}{p^2}.$

(ii) $p^2 - 4p + 2 + \frac{4}{p} + \frac{1}{p^2}.$

(iii) $p^2 - 2p + 3 - \frac{2}{p} + \frac{1}{p^2}.$

(iv) $\frac{a^2}{p^2} + \frac{2a}{p} - 1 - \frac{2p}{a} + \frac{p^2}{a^2}.$

(v) $p^4 - 2p^3 + \frac{3p^2}{2} - \frac{p}{2} + \frac{1}{16}.$

(vi) $\frac{4a^2}{9b^2} + \frac{a}{b} + \frac{391}{240} + \frac{6b}{5a} + \frac{16b^2}{25a^2}.$

EXERCISE XXXV.

ALGEBRAIC DIVISION.

A.

1. Obtain the algebraic quotients in the following cases :—

(i) $(a^3 - 6a^2 + 16a - 21)/(a - 3)$.
(ii) $(3a^3 + 29a^2b + 64ab^2 - 12b^3)/(a + 6b)$.
(iii) $(2a^3 - 19a^2x^2 + 13ax^4 + 55x^6)/(2a - 5x^2)$.
(iv) $(a^4 - 7a^3 + 13a^2 - 19a + 20)/(a - 5)$.
(v) $(2 + 13p + 20p^2 + 5p^3 + 24p^4)/(1 + 3p)$.
(vi) $(9a^4 + 6a^3 - 21a^2 - 2a + 8)/(3a + 2)$.
(vii) $(21 - 25x - 4x^2 - 27x^3 + 36x^4)/(3 - 4x)$.
(viii) $(a^4 - 7a^2 + 3a + 6)/(a - 2)$.
(ix) $(6a^5 - 23a^4 + 40a^2 + 49)/(3a - 7)$.
(x) $(1 - 21a^3 - 54a^5)/(1 - 3a)$.

2. Divide :—

(i) $a^3 - a^2 - 9a - 12$ by $a^2 + 3a + 3$.
(ii) $6p^3 - p^2q - 14pq^2 + 3q^3$ by $3p^2 + 4pq - q^2$.
(iii) $n^4 - n^3 - n + 1$ by $n^2 + n + 1$.
(iv) $2a^5 + 3a^4 - 12a^3 + 15a^2 - 11a + 3$ by $2a^2 - 3a + 1$.
(v) $p^5 - 4p^4 + 3p^3 + 3p^2 - 3p + 2$ by $p^2 - p - 2$.
(vi) $6a^5 - 2a^4 - 5a^3 + 18a^2 - 11a + 10$ by $3a^2 - a + 2$.
(vii) $2a^6 - 3a^5 + 4a^4 + 7a^3 - 9a^2 + 14a + 21$ by $a^3 + 2a + 3$.
(viii) $a^5 - 3a^3 + 9a - 3$ by $a^2 - 3a + 1$.
(ix) $2 - 6x + x^2 + 15x^3 - 12x^4 - 9x^5 + 9x^6$ by $1 - 3x + 3x^2$.
(x) $1 - 16x^5 + 3x^6$ by $1 - x + 3x^2$.

3. Find the integral equivalents of the following fractions :—

(i) $\dfrac{a^4 - 1}{a + 1}$. (ii) $\dfrac{a^4 - b^4}{a + b}$. (iii) $\dfrac{a^5 + 1}{a + 1}$.
(iv) $\dfrac{p^6 - q^6}{p + q}$. (v) $\dfrac{1 - a^9}{1 - a}$. (vi) $\dfrac{1 + a^{13}}{1 + a}$.

Verify the results by putting $a = +4$ in (i), $a = -2$ in (iii), $a = \frac{1}{10}$ in (v).

4. Find out why there are no integral equivalents of the fractions $(a^4 + 1)/(a + 1)$, $(1 + a^5)/(1 - a)$, $(1 + a^6)/(1 - a)$.

5. Explain (i) why $(1 + a^n)/(1 + a)$ has an integral equivalent when n is odd but not when n is even; (ii) why $(1 - a^n)/(1 + a)$ has an integral equivalent when n is even

but not when it is odd; (iii) why $(1 - a^n)/(1 - a)$ always has an integral equivalent; (iv) why $(1 + a^n)/(1 - a)$ never has an integral equivalent. (It is assumed that n is a positive whole number.) Write down the equivalents where they exist.

B.

Note.—The expression $a^4b - 3a^2b^3 - 4c^2$ is said to be of the **fourth degree** in a, of the **third degree** in b, and of the **second degree** in c; the degree being measured by the highest power of the variable named. The expression is also said to be of the **fifth degree** in a and b, since in terms in which both these variables occur together the sum of their powers is five.

6. Find the values of P and Q in the following identities with the restriction that the degree of Q is to be as low as possible:—

(i) $a^2 - 4a + 7 = (a + 2)P + Q$.
(ii) $2a^2 + 3a - 1 = (a - 3)P + Q$.
(iii) $a^3 - 3a^2 + 2a - 1 = (a + 1)P + Q$.
(iv) $4a^3 - 7a + 3 = (2a - 5)P + Q$.
(v) $a^4 + 3a^2 - 4 = (a^2 - 7)P + Q$.
(vi) $a^4 + 2a^3 - a + 1 = (a + 4)P + Q$.

7. From your result in No. 6 (i) fill in the values of P and Q in the identity,

$$\frac{a^2 - 4a + 7}{a + 2} = P + \frac{Q}{a + 2}.$$

In other words, write down the **integral expression** and the proper fraction (or **complement**) which are together equivalent to the fraction $(a^2 - 4a + 7)/(a + 2)$.

8. Use the results of No. 6 to express each of the following fractions as equivalent to an integral expression together with a complementary proper fraction:—

(i) $\dfrac{a^3 - 3a^2 + 2a - 1}{a + 1}$. (ii) $\dfrac{4a^3 - 7a + 3}{2a - 5}$.

(iii) $\dfrac{a^4 + 2a^3 - a + 1}{a + 4}$.

9. What is the value (i) of $(a - 3)P$ when $a = 3$; (ii) of $(a + 2)P$ when $a = -2$; (iii) of $(2a - 5)P$ when $a = \frac{5}{2}$? Do the answers depend at all on the value of P?

10. Turn to your working of No. 6 (i). What is the value of the right-hand side when $a = -2$? What, then, should be the result of substituting -2 for a in the left-hand side? Find if it is so.

11. What rule would you give for finding Q *without previously finding* P in problems like those of No. 6? Test the rule by seeing whether it gives you the values for Q which you have already obtained in (ii), (iii), and (v).

12. Find the complementary proper fraction in the following cases without finding the quotient :—

(i) $(a^3 - 4a^2 + 6a + 2)/(a + 3)$.
(ii) $(a^3 - 7a^2 + 13a - 4)/(a - 4)$.
(iii) $(a^4 + a + 1)/(a + 1)$.
(iv) $(a^5 - 2a^3 + 3a^2 - 4)/(a + 1)$.
(v) $(2a^4 - 5a^2 - a + 3)/(2a - 3)$.

13. Find the values of P and Q in the following cases, the degree of Q being as low as possible :—

(i) $a^4 - a^3 + 2a + 1 = (a^2 - 2a + 3)P + Q$.
(ii) $3a^4 + a^3 + 5a^2 - 5a + 2 = (3a^2 - 2a + 1)P + Q$.
(iii) $6a^4 - 5a^3b + 4a^2b^2 - 5ab^3 + 6b^4 = (a^2 - 2ab - 7b^2)P + Q$.
(iv) $a^5 - 1 = (a^2 + a - 1)P + Q$.

14. Use the results of No. 13 to exhibit each of the following fractions as equivalent to an integral expression together with a proper fraction :—

(i) $(a^4 - a^3 + 2a + 1)/(a^2 - 2a + 3)$.
(ii) $(3a^4 + a^3 + 5a^2 - 5a + 2)/(3a^2 - 2a + 1)$.
(iii) $(a^5 - 1)/(a^2 + a - 1)$.

15. Arrange $a^4 - a^3 - 2a^2 - pa + 3$ in the form

$$(a^2 - 3a + 1)P + Q.$$

16. For what value of p is $a^4 - a^3 - 2a^2 - pa + 3$ exactly divisible by $a^2 - 3a + 1$?

17. For what value of p is $a^5 - a^4 + pa + 8$ exactly divisible by $a^2 + a + 2$?

C.

Note.—When the integral part of the equivalent of a fraction is calculated in ascending powers of the variable (e.g. $1 + 2a - 3a^2 + 4a^3 +$ etc.) the complement cannot be a proper fraction; for the degree of the numerator will be higher the greater the number of terms in the integral part. The integral part is in these cases often called the **expansion** of the fraction in ascending powers of the variable.

18. Find the integral expression and the complementary fraction which are equivalent to the fractions

(i) $(1 + a^4)/(1 + a)$. (ii) $(1 - a^7)/(1 + a)$.
(iii) $(1 + a^6)/(1 - a)$. (iv) $(1 + a^9)/(1 - a)$.

19. What is the value of the complementary fraction in No. 18 (i) when $a = \frac{1}{10}$?

20. What is the numerical value of the error that would be made by assuming

$$(1 - a^7)/(1 + a) = 1 - a + a^2 - \ldots + a^6$$

when $a = \frac{1}{3}$? What is the ratio of the error to the whole value of the fraction?

21. Within what degree of numerical accuracy can we write

$$\frac{(1 + a^9)}{1 - a} = 1 + a + a^2 + \ldots + a^8$$

when $a = \frac{1}{2}$?

22. Why are the questions in Nos. 19-21 asked with regard to values of a less than 1?

23. Show that

$$\frac{1}{1 - a} = 1 + a + a^2 + a^3 + a^4 + \frac{a^5}{1 - a}.$$

Show that the ratio of the complement to the whole value of the fraction is a^5. Putting $S \equiv$ the complete value of the fraction, show that the error involved in neglecting the complement after the term a^4 is less than $S \times 10^{-3}$ when $a = \frac{1}{4}$.

24. Show that, in general,

$$\frac{1}{1 - a} = 1 + a + a^2 + \ldots + a^{n-1} + \frac{a^n}{1 - a}$$

and that the complement, after the fraction has been expanded to n terms, is $a^n.S$; S being the complete value of the fraction. Show also that if a is numerically less than 1 the complement can be made as small as is required by taking n sufficiently large.

25. Expand the following fractions to the number of terms indicated, adding in each case the complement after the expansion :—

(i) $1/(1 + a)$ to a^5.
(ii) $1/(1 + a^2)$ to 7 terms.
(iii) $1/(1 - 2a)$ to 6 terms.
(iv) $1\Big/\left(1 + \frac{a^2}{3}\right)$ to 5 terms.

(v) $(1 - 2a)/(1 - a)$ to a^6.
(vi) $(1 - a)/(1 + 2a)$ to 5 terms.
(vii) $(1 + a)/(1 + \frac{1}{2}a)$ to 7 terms.
(viii) $(1 - \frac{1}{2}a)/(1 + \frac{1}{3}a)$ to 4 terms.

Note.—We have
$$\frac{3}{1-a} = 3 \times \frac{1}{1-a}$$
$$= 3(1 + a + a^2 + \ldots).$$

Also
$$\frac{1}{2-a} = \frac{1}{2(1 - \frac{1}{2}a)}$$
$$= \frac{1}{2}\frac{1}{1 - \frac{1}{2}a}.$$

26. Give the specified number of terms of the expansion and the complement:—

(i) $2/(1 - 3a)$ to 4 terms.
(ii) $3/(1 + \frac{1}{2}a)$ to 5 terms.
(iii) $1/(2 - a^2)$ to 6 terms.
(iv) $1/(3 + 2a)$ to 4 terms.
(v) $(1 - 2a)/(2 + a)$ to 5 terms.
(vi) $(1 + \frac{1}{2}a)/(4 - 3a)$ to 4 terms.

27. Show that
$$\frac{1}{1 + a + a^2} = 1 - a + a^3 - a^4 + \frac{a^6}{1 + a + a^2}.$$

28. Expand $1/(1 + 2a - a^2)$ as far as the fourth term, and show that the complement is $(29a^4 + 12a^5)/(1 + 2a - a^2)$.

29. Find the first four terms of the expansion of
$$(1 + 2a)/(1 - a + a^2)$$
and the complement after the expansion.

30. Find the first five terms of the expansion of $1/(1 + x)^2$. What is the complementary fraction?

EXERCISE XXXVI.

GEOMETRIC SERIES.

A.

1. A lamp hanging at the end of a chain is pulled to the right 1 metre out of the vertical and is then released. It swings to a point 0·9 m. to the left of the vertical, then to a point 0·81 m. to the right of the vertical. The succeeding swings diminish in accordance with the same law. Find the greatest distance the lamp could travel before coming to rest (i) including the first movement of 1 m. to the right, (ii) excluding this movement and counting only the free swinging of the lamp. Show in case (i) that when the lamp passes through the vertical for the sixth time after being withdrawn it has accomplished more than half its greatest total movement. Illustrate by a diagram, showing the total space travelled after successive swings.

2. A ball is lifted 10 feet from the floor and is then dropped. It bounces to a height of 8 feet. Each subsequent bounce carries it four-fifths as high as the preceding one. Find the greatest distance the ball could travel before coming to rest (i) including and (ii) excluding the distance through which it was lifted in the first instance. After how many bounces will the ball have travelled altogether one-half of the maximum distance? After how many bounces three-quarters of the maximum distance?

3. A weight is hanging at the end of a vertical rubber cord. I pull it down 10 inches and release it. It swings up and down for a considerable period, the amplitude of each semi-vibration being 0·7 of the preceding one. (The amplitude is the extreme distance the weight rises or sinks above or below the position of rest. Each vibration includes two amplitudes—one below and one above.) Calculate the distance the weight would travel if it vibrated according to this law for ever. Show that after three semi-vibrations it has actually travelled more than 75 per cent. of this distance.

When will it have completed 90 per cent? Illustrate your solution by a diagram.

4. A man undertook to pay £1000 to a charity one year, £750 next year, three-quarters of this sum in the third year, and so on until his death. What is the outside limit of the expectations of the charity? If the man died after making twenty donations, how much would their total fall short of this sum? $[(\frac{3}{4})^{20} = 0{\cdot}003171.]$

5. Another subscriber to the same charity promised to give £100 the first year and to increase his donation by one-tenth every year as long as he lived. What would be the total of his donations if he also lived for twenty years? What would be the amount of his last donation? $[(1{\cdot}1)^{20} = 6{\cdot}727.]$ Illustrate Nos. 4 and 5 by two parallel strips in which the first and last donations are marked off on the same scale and the limiting sum is indicated.

B.

Note.—A series of terms each of which is obtained by multiplying its predecessor by a constant factor is called a **geometric sequence**. The constant factor is called the **common ratio** of the sequence. It is evident that, unlike an arithmetic sequence, a geometric sequence can be continued without end both ways even if the terms are non-directed numbers.

A series of numbers made by taking any term of a geometric sequence and any number, limited or unlimited, of consecutive terms which immediately follow it is said to be in, or to form, a **geometric progression** (G.P.). [*Cf.* Ex. XXIX, C, Note.]

We have seen that when the common ratio is positive and numerically less than unity the sum of a G.P. never goes beyond a certain value however many terms are taken, although it can be made, by increasing the number of terms, to approach, and thereafter to keep, as near to that value as we please. This value is best called the **limiting sum** of the G.P., but is more generally known as "the sum to infinity". [See Ex. XXXV, Nos. 22, 24.] When the ratio is negative and numerically less than unity the sum of n terms also constantly approaches nearer to the limiting sum as n increases but swings alternately above and below it.

Each of the terms between the first and last term of a G.P. is called a **geometric mean.**

6. State the common ratio of each of the following sequences. Continue each for three terms both ways :—

(i) . . . $4, 6, 9, 13{\cdot}5, \ldots$

(ii) . . . $-8, +6, -4{\cdot}5, \ldots$

(iii) . . . $\frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$

(iv) . . . $+\frac{p^2}{q^3}, -\frac{p}{q^2}, +\frac{1}{q}, \ldots$

(v) . . . $a, \frac{1}{a-b}, \frac{1}{a(a-b)^2}, \ldots$

7. In No. 6 (i) give expressions for the 13th term to the right of 6 and the 15th to the left of 13·5 in terms of positive and negative powers of the common ratio.

8. In No. 6 (ii) give similar expressions for the 8th term to the right and the nth term to the left of $+6$. [The symbol $(-)^n$ is used to denote the sign of a product of n negative factors.]

9. In No. 6 (iv) give similar expressions for the 9th term to the right of $+\frac{1}{q}$ and the n term to the left of $-\frac{p}{q^2}$.

10. In No. 6 (v) give similar expressions for the nth terms before and after a.

11. Find the limiting sum (the "sum to infinity") of each of the following series :—

(i) $S = 1 + \frac{1}{2} + \frac{1}{4} + \ldots$

(ii) $S = +1 - \frac{1}{2} + \frac{1}{4} - \ldots$

(iii) $S = 24 + 20 + 16\frac{2}{3} + \ldots$

(iv) $17{\cdot}5 + 5{\cdot}25 + 1{\cdot}575 + \ldots$

(v) $1 - \frac{a}{b} + \frac{a^2}{b^2} - \ldots \quad [\,|a/b| < 1].$

(vi) $\frac{(a+b)^2}{(a-b)^2} + \frac{a+b}{a-b} + 1 + \ldots \quad [\,|(a+b)| > |(a-b)|\,].$

Illustrate the summation of (i) and (ii) by two parallel strips marked off as in No. 5.

Note.—The symbol $|a|$ means the numerical value of a, i.e. its value when its sign is removed.

12. Write formulæ giving the difference between the sum of n terms of No. 11 (i), (iv), and (v) and the limiting sum.

13. Let $S_n \equiv$ the sum of n terms of a G.P. and $S \equiv$ the limiting sum. It is often convenient to write

$$S - S_n = kS.$$

Find the value of k in No. 11 (i), (iii), and (vi).

14. Write a formula for the sum of n terms of each of the following series:—

(i) $1 + 2 + 4 + 8 + \ldots.$

(ii) $100 + 102 + 104{\cdot}04 + \ldots.$

(iii) $-8 + 10 - 12{\cdot}5 + \ldots.$

(iv) $\frac{1}{81} + \frac{1}{72} + \frac{1}{64} + \ldots.$

(v) $q + p^2 + \frac{p^4}{q} + \ldots$ $[p^2 > |q|]$.

15. In a G.P. let $a \equiv$ the first term, $r \equiv$ the common ratio, $n \equiv$ the number of terms. Write formulæ for (i) the nth term; (ii) the sum of n terms, r being greater than unity; (iii) the same, r being less than unity; (iv) the limiting sum in the last case.

16. Write out full proofs of the formulæ of No. 15 (ii) and (iv). Prove by means of your formulæ the properties described in the third paragraph of the note to No. 6.

C.

17. A sum of P pounds is invested at compound interest for n years, the rate of interest being 3 per cent per annum. Show that its amount, A, is given by the formula

$$A = P \times (1{\cdot}03)^n.$$

Change the subject of this formula to P, using a negative index for the sake of conciseness. How will you describe P as used in the second formula?

18. Compound interest is given at the rate of i per *pound* per annum. Write formulæ for A, the amount to which a sum P would accumulate in n years, and for P, the sum which would, by accumulation, produce A in n years. (Note that P is called the **present value** of A, A the **amount** of P.)

19. On 1 January, 1905, a man, X, determined to save £20 every year for the next five years and to invest it at the end of each year in a business that promised him 3 per cent per annum compound interest. Accordingly he invested £20 on 1 January, 1906, 1907, 1908, 1909, 1910. Show that the total sum standing to his credit immediately after he paid in the last £20 was

$$A = £20 \{ (1{\cdot}03)^4 + (1{\cdot}03)^3 + (1{\cdot}03)^2 + (1{\cdot}03) + 1 \}$$
$$= £\frac{2000}{3}\{(1{\cdot}03)^5 - 1\}.$$

According to the Compound Interest Tables £1 will, at 3 per cent compound interest, accumulate in five years to £1·1593. Use this number to calculate the value of A.

20. Another man, Y, invested in the same business on 1 January, 1905, a single sum which by 1 January, 1910, had accumulated to exactly the same amount as X's successive investments in No. 19. Show that this single sum was

$$P = £\frac{2000}{3} \times \frac{(1{\cdot}03)^5 - 1}{(1{\cdot}03)^5}.$$

Calculate its amount to the nearest tenth of a pound.

Note.—Suppose that Y, instead of investing £91·6 in the business on 1 January, 1905, had, on that date, lent this sum to X. Suppose, further, that X undertook to return the loan with interest by paying Y £20 on 1 January, 1906, 1907, 1908, 1909, 1910. Finally, suppose that Y on the days when he received each instalment invested it as X did in No. 19. Then the answers to Nos. 19 and 20 show that on 1 January, 1910, Y would have been in exactly the same position as he would have been if he had invested his money as in No. 20. In other words, to lend £91·6 and to receive in return five annual payments of £20 is financially equivalent to investing £91·6 for five years at 3 per cent.

A number of equal payments made at regular intervals in consideration of a lump sum previously received is called an **annuity**. The lump sum is called the **present value** or the **cost** of the annuity. The sum paid periodically is called the **rent** of the annuity. The relation between the rent and the cost is fixed not only by the number of payments (the **term** of the annuity) but also by the rate of interest expected by the person who buys the annuity or makes the loan which is to be repaid by the annuity.

21. Let $P \equiv$ the present value, $A \equiv$ the amount, $a \equiv$ the rent, $n \equiv$ the term of an annuity, and let $i \equiv$ the interest to be earned by £1 in the interval between two payments. Show that

$$A = a \cdot \frac{(1+i)^n - 1}{i},$$

and that

$$\begin{aligned} P &= A(1+i)^{-n} \\ &= a \cdot \frac{(1+i)^n - 1}{i(1+i)^n} \\ &= a \cdot \frac{1-(1+i)^{-n}}{i}. \end{aligned}$$

22. Convert the last two formulæ into formulæ for finding the rent of an annuity, given the present value, etc.

23. A man borrows £500 from a building society in order to buy a house. The loan is to be returned with interest at 4 per cent by ten equal annual payments. The first payment is to be made on the first anniversary of the loan. Calculate the annual payment. [The amount of £1 in ten years at 4 per cent compound interest is £1·4802.]

24. Another man borrows £500 upon the same terms, except that the re-payment is to be spread over fifteen years. Calculate the amount to be paid annually. [In fifteen years £1 becomes £1·8009 at 4 per cent compound interest.]

25. The Urban District Council of a seaside town invite subscriptions to a loan of £21,000 for the construction of a new pier. The loan is to be discharged by seven equal annual amounts paid out of the rates, interest at $3\frac{1}{2}$ per cent being allowed. What annual charge upon the rates will be required? [The amount of £1 for seven years at $3\frac{1}{2}$ per cent compound interest is £1·2723.]

26. A man buys for £800 a house of which the lease has forty-three years to run. The property is subject to a ground rent of £10 per annum. (This means that for the next forty-three years the man or his successors will receive the rent of the house but must pay out of it £10 to the owner of the land.) By a special arrangement the rent and ground rent are paid once a year on the anniversary of the purchase. Calculate the amount of the rent if the investment is to yield 5 per cent. [Amount of £1 for forty-three years at 5 per cent compound interest = £8·1497.]

27. Show by means of the last formula of No. 21 that no matter how long an annuity runs its cost cannot exceed a limiting value given by the formula

$$P = a/i.$$

28. After the battle of Trafalgar, Parliament made a grant of £6000 per annum to Nelson's heir and his descendants for ever. What single sum would have been equivalent to this grant, interest being reckoned at 3 per cent per annum?

29. In ancient times a settled proportion of the produce on agricultural land was set aside for the maintenance of the clergy. It was called the **tithe of produce.** Since 1836 the tithe has been paid in money. When land is taken for building purposes, etc., the tithe is often **redeemed,** that is, the tithe-owner surrenders his right to receive annual tithes for ever in consideration of an equivalent lump sum paid to him by the landowner. The interest assumed in calculating the redemption must not be more than 4 per cent.

The tithe upon a certain piece of property is £3 10s. per annum. What sum is needed to redeem it, interest being reckoned at $3\frac{1}{2}$ per cent?

Note.—The formula of No. 25 shows that a perpetual annuity can be purchased by $1/i$ times the annual rent. For this reason $1/i$ is called the **number of years' purchase.** Thus if a tithe were redeemed or a freehold property bought for twenty-five years' purchase the interest would be 4 per cent.

30. A man takes a piece of land on a lease for 999 years at an annual rent of £24. After a short time the landowner agrees to let him have the freehold of the property for thirty years' purchase. How much must the purchaser pay for the land and at what rate is interest reckoned?

EXERCISE XXXVII.

THE COMPLETE NUMBER SCALE.

Note.—The "scale" required in some of the examples is a long straight line graduated uniformly with *plus* or *minus* numbers from an origin in the middle of its length. The graduation should be carried to $\pm$ 50.

A.

1. Show on a scale the various positions of a point P which represents in turn the values of a, $a + b$, $a - b$, ab, a/b, ab^2 first when $a = -2$, $b = +5$, and secondly when $a = +2$, $b = -5$. (The successive positions should be marked P_0, P_1, P_2, etc.)

2. Show on a scale the movements of a point P which marks the successive values of ab^n (i) when $a = -1$, $b = +2$; and (ii) when $a = +20$, $b = -\frac{1}{2}$, while n assumes in succession the values 0, + 1, + 2, . . . + 5 in each case.

3. Repeat the two investigations of No. 2, giving n in succession the values 0, − 1, − 2, . . . − 5 in each case.

4. Mark on a scale (where possible) the positions of a point which registers the values of $a(b + pc^n)$ when $a = -5$, $b = +4$, $c = -\frac{1}{3}$, $p = -9$ and n assumes in succession the values − 4, − 3, − 2, − 1, 0, + 1, + 2, + 3, + 4. What will happen as the value of n continually rises? What would happen if the value of n were continually lowered?

5. A point P occupies in succession the points 0, + 10, + 20, . . . + 50. Label these points P_0, P_1, etc., upon your scale and label with the letters Q_0, Q_1, Q_2, etc., the approximate positions of all the square roots of these numbers.

6. Indicate with suitable labels the numbers − 50, − 40, − 30, . . . 0, + 10 . . . and the approximate position of their cube roots.

Note.—The symbol ∞ is used to denote a very large number—that is one whose reciprocal is very nearly zero.

7. Indicate on a scale the range of values assumed by the expression $(15 + 30x) / (1 + 3x)$ as x moves from $-\infty$

through zero to $+\infty$. Move your pencil point along the range in the way in which the representative point would move as x passes through its various values. (Before calculating values transform the expression as in Ex. XXXV, No. 7.)

8. How would the results of No. 7 have been different if the expression had been $(15 + 30x)/(1 - 3x)$?

9. Repeat the investigation of No. 7 upon the values of $(15 + 30x^2)/(1 + 3x^2)$, as x moves from $-\infty$ to $+\infty$.

10. The point P marks the values of the expression $\frac{1}{x} - 4x$. Show that it will traverse the whole scale of numbers twice in the negative direction while x traverses it once in the positive direction.

B.

11. Find the value of the abstract variable x given that

$$\tfrac{1}{3}(2x + 17) - 7 = 10 - \tfrac{1}{2}(1 - 3x).$$

Let two points P and Q record on parallel scales the values of the left and right-hand sides of this relation. Indicate (by the letters P_0, P_1, etc., and Q_0, Q_1, etc.) the positions of P and Q at the different stages of the solution and so justify it.

12. Solve the equation $\dfrac{3x}{2x + 7} - 5 = 0$. Justify the various stages of the solution by the method of No. 11.

13. Solve the following equations:—

(i) $\frac{1}{2}(2 - x) - \frac{1}{5}(5x + 21) - (x + 3) = 0$.

(ii) $\dfrac{x + 3}{5} - \dfrac{6 - x}{10} + (0{\cdot}7 - x) = 0$.

(iii) $(2x - 1)(2x + 3) - (4x + 7)(x - 2) = 0$.

(iv) $8 - 2\{5x - 7(4 + 3x)\} = 0$.

(v) $\sqrt{(2x - 7)} + 6 = 0$.

(vi) $\sqrt{(2x - 7)} - 6 = 0$.

(vii) $\sqrt[3]{(5x + 8)} + 3 = 0$.

(viii) $3 - \sqrt[3]{(5x - 8)} = 0$.

(ix) $2x - \sqrt{\{(4x + 1)(x - 4)\}} - 3 = 0$.

(x) $(3x - 1)^2 - (x - 3)^2 = 0$.

(xi) $(x - 2)^3 - x(x - 4)(x - 5) + 4(x - 1) = 0$.

(xii) $(2x + 1)(4x^2 - 4x + 1) + (x + 2)(x^2 - 2x + 4) = 0$.

14. Express the sum of the two fractions $P/(x - 1)$ and $Q/(x - 2)$ as a single fraction.

15. What is the numerator of the fraction obtained in No. 14 if $P = +3$ and $Q = -1$?

16. What must be the values of P and Q in order that the numerator may be (i) $+1$, (ii) x, (iii) $2x - 3$?

17. Use the results of No. 16 to calculate the value of

$$\frac{2x - 3}{x^2 - 3x + 2} \text{ and } \frac{x}{x^2 - 3x + 2},$$

(i) when $x = 0$, (ii) $x = +4$, (iii) $x = -3$.

Note.—The fractions $1/(x - 1)$, $1/(x - 2)$ are called the **partial fractions** of the fraction

$$(2x - 3)/(x^2 - 3x + 2).$$

Similarly $-1/(x - 1)$ and $2/(x - 2)$ are the partial fractions of $x/(x^2 - 3x + 2)$.

18. Use the method of Nos. 14 and 16 to find the partial fractions of the following:—

$$\text{(i) } \frac{x}{x^2 + 5x + 6}; \quad \text{(ii) } \frac{5x - 2}{x^2 - 4};$$

$$\text{(iii) } \frac{7(2 - x)}{2x^2 + x - 6}; \quad \text{(iv) } \frac{9x + 52}{12x^2 + 13x - 14}.$$

19. Use your results to find the value of (i) when $x = -4$, of (ii) when $x = +8$, of (iii) when $x = +7$, and of (iv) when $x = -13$.

20. Find values of P and Q to satisfy the relation.

$$\frac{2x + 1}{(x - 1)^2} = \frac{P}{x - 1} + \frac{Q}{(x - 1)^2}.$$

Verify your result by putting $x = +4$ and $x = -9$.

21. Analyse the following into partial fractions after the pattern of No. 20.

$$\text{(i) } \frac{2x + 1}{(x + 3)^2}; \quad \text{(ii) } \frac{2x - 9}{(2x - 3)^2};$$

$$\text{(iii) } \frac{9x + 7}{(3x + 4)^2}; \quad \text{(iv) } \frac{2(x^2 + 1)}{(x + 2)^3}.$$

Verify each answer by substituting a positive or negative value for x.

22. Express each of the following fractions as the sum of an integral expression and a series of partial fractions.

$$\text{(i) } \frac{6x^2 - 10x + 16}{2x^2 + 3x - 2}; \quad \text{(ii) } \frac{9x^2 - 29x + 38}{3x^2 + x - 10};$$

$$\text{(iii) } \frac{4(x^2 - 2x + 3)}{4x^2 - 12x + 9}; \quad \text{(iv) } \frac{x(4x^2 + 9x + 8)}{(x + 1)^3}.$$

Verify any two of your answers by substitution.

23. Reduce to single fractions (i) $\dfrac{1}{x - 10} - \dfrac{1}{x - 7}$ and

(ii) $\dfrac{1}{x-9} - \dfrac{1}{x-6}$.

Use your results to find the value of x which satisfies the condition :—

$$\frac{1}{x-10} - \frac{1}{x-7} = \frac{1}{x-9} - \frac{1}{x-6}$$

Also use them to find a value of x such that

$$\frac{x-9}{x-10} - \frac{x-6}{x-7} = \frac{x-8}{x-9} - \frac{x-5}{x-6}.$$

[Replace each fraction by its equivalent integral expression and complementary fraction. The relation then becomes identical with the former one.]

24. Find the values of the variable which comply with the following conditions :—

(i) $\dfrac{x-3}{x-4} - \dfrac{2x-3}{2x-5} = \dfrac{3}{2x^2-6}$.

(ii) $\dfrac{x}{x-2} - \dfrac{x+1}{x-1} = \dfrac{x-8}{x-6} - \dfrac{x-9}{x-7}$

(iii) $\dfrac{2x-13}{x-1} + \dfrac{3x+5}{x-2} = \dfrac{4x+23}{x+3} + \dfrac{x-7}{x+4}$.

C.

25. If $(7 - 3x)/4 < 0$ what can be said about the value of x?

Justify your argument (as in Nos. 11 and 12) by means of a pair of number-scales.

26. Show that if x is positive $3x/(4x^2 - 9) > 0$ only if $x > +\frac{3}{2}$, and that if x is negative $3x/(4x^2 - 9) > 0$ only if $x < -\frac{3}{2}$.

27. Show that $3x/(4x^2 - 9) < 0$ only if $+\frac{3}{2} > x > -\frac{3}{2}$. Mark on a scale the ranges of values of x that make $3x/(4x^2 - 9) > 0$ and < 0 respectively.

28. Mark on a scale the range of values of x within which

(i) $(7 - 3x)/(x - 6) > 0$; (ii) $(22 + 7x)/(8x - 50) < 0$;
(iii) $(x - 7)(x + 2) > 0$; (iv) $(x - 4)(x + 8) < 0$;
(v) $x^2 - 5x - 8 < -14$; (vi) $6x^2 + 17x + 5 > +19$.

29. There are two directed numbers m and n. When the first is multiplied by $+4$ and the second by -5 the algebraic sum of the products is $+23$. When the first is multiplied by -7 and the second by $+4$ the algebraic sum of the products is -7. Calculate the values of m and n by the composition method, and justify your procedure.

30. Find the values of directed numbers which satisfy simultaneously the following sets of conditions :—

(i) $15m + 19n = 18,\ 19m + 15n = 50.$

(ii) $3m - 4n + 2 = 5m - 6n - 2 = 7m + 2n + 4.$

(iii) $\frac{4}{m} - \frac{2}{n} = -7;\ \frac{3}{m} + \frac{5}{n} = +10.$

(iv) $\frac{6}{m} + \frac{2}{n} = -1;\ 2mn - 3m + 5n = 0.$

(v) $(m + 1)(n + 5) = (m + 5)(n + 1),\ mn + m + n = (m + 2)(n + 2).$

(vi) $m + n = 1;\ (2m - 3)(8n + 1) = (4m - 5)(4n + 5).$

(vii) $m + n + p = -1,\ 2m + 3n + p = +2,$
$4m + 9n + p = +14.$

(viii) $m - 2n = -3,\ m^2 - 4n^2 = +12.$

(ix) $m + n = -4,\ mn = -21.$

(x) $m - n = -25,\ 4m - 4n = mn.$

EXERCISE XXXVIII.

FURTHER EXAMPLES ON DIRECTED NUMBERS.

A.

1. Given that $t = \dfrac{W}{W + w}\left(1 + \dfrac{V}{v}\right) \cdot \dfrac{2a}{v}$
calculate t when $W = 190$, $w = 2$, $V = -4$, $v = +100$, $a = 0{\cdot}1$.

2. Calculate the value of

$$\frac{(1 + px)(1 + qx^2)}{1 + rx^3}$$

(i) when $x = 0$, (ii) when $x = -10$, (iii) when $x = +10{,}000$, (iv) when $x = -\infty$; given that $p = -\frac{1}{2}$, $q = +\frac{1}{4}$, $r = -\frac{1}{5}$.

3. Find the value of

$$a^x \cdot \frac{1 + b \sin \alpha}{1 + c \cos \alpha}$$

(i) when $x = 0$, $\alpha = 12°$, (ii) when $x = -4$, $\alpha = 55°$; given that $a = +10$, $b = -5$, $c = +4$.

4. Evaluate

$$1 + \frac{a+b}{a+2b} \cdot \frac{x}{p-q} + \left(\frac{a+b}{a+2b}\right)^2 \cdot \frac{x^2}{p^2 - q^2} + \left(\frac{a+b}{a+2b}\right)^3 \cdot \frac{x^3}{p^3 - q^3}$$

(i) when $x = 0$, $p = +2$, $q = -3$, (ii) when $x = +10$, $p = -3$, $q = +2$, (iii) when $x = -10$, $p = -3$, $q = +2$; given that $a = +3$, $b = -2$.

5. In the formula

$$\tan(\alpha - \beta) = \frac{2^{px} - 2^{-px}}{2^{px} + 2^{-px}}$$

$\beta = 20°$ and $p = +\frac{1}{5}$. Find the value of α (i) when $x = 0$, (ii) when $x = +10$. Prove also (iii) that when x is positive and large α is practically constant with a value of $65°$.

B.

6. Express $\quad a - \dfrac{1}{1 - a} + \dfrac{1 - 3a + a^3}{1 - a^2}$

as a single algebraic fraction. Show without further calculation that the expression

$$a + \frac{1}{a+1} + \frac{1 + 3a - a^3}{a^2 - 1}$$

has the same value. Verify your conclusion by substituting (i) $a = 0$, (ii) $a = +5$, (iii) $a = -7$, in each of the original expressions.

7. Simplify the product

$$\left(1 - \frac{x-2}{x^2+x-2}\right) \cdot \frac{x+2}{x}.$$

What new identity can be deduced from your result by substituting $x + 1$ for x? Verify both identities by putting (i) $x = +2$, (ii) $x = -\frac{1}{2}$.

8. Reduce each of the following expressions to a single algebraic fraction in its lowest terms :—

(i) $\dfrac{(a-2b)^2}{(a+2b)^2} - \dfrac{(a+2b)^2}{(a-2b)^2}$.

(ii) $\dfrac{1}{a+2b} + \dfrac{1}{a-2b} - \dfrac{8b^2}{a^3-4ab^2}$.

(iii) $\dfrac{1}{x^2-1} + \dfrac{1}{x^2+1} - \dfrac{x^2-1}{x^4+x^2+1} - \dfrac{x^2+1}{x^4-x^2+1}$.

(iv) $1 + \dfrac{2x-3}{x-1} + \dfrac{x-1}{(x+1)^2} + \dfrac{4x}{(x+1)^2 \cdot (x-1)}$.

9. Show that the addition of the term $(p + 1)^2$ to the expression

$$\tfrac{1}{6}p(p+1)\,(2p+1)$$

produces the same result as the substitution of $p + 1$ for p. Verify by putting (i) $p = +9$, (ii) $p = -6$.

10. Show that the addition of the term $(p + 1)^3$ to the expression

$$\tfrac{1}{4}(p+1)^2p^2$$

produces the same result as the substitution of $p + 1$ for p.

Verify by putting (i) $p = +2$, (ii) $p = -11$.

11. Show that

$$\begin{aligned}(1 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r)/(1 - x) &= \\ = 1 + (1+a_1)x + (1+a_1+a_2)x^2 + (1+a_1+a_2+a_3)x^3 &+ \ldots \\ \ldots + (1 + a_1 + a_2 + \ldots + a_r)x^r &+ \frac{a_rx^{r+1}}{1-x}.\end{aligned}$$

12. Use the identity of No. 11 to prove that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ . \ . \ .$$

$$. \ . \ . + (r+1)\,.\,x^r + \frac{(r+1)\,.\,x^{r+1}}{1-x} + \frac{x^{r+1}}{(1-x)^2}.$$

[Start from the expansion of $1/(1-a)$ proved on p. 199.]

13. Deduce from this equivalence formulæ for calculating $1/(1-x)^2$ (i) to a first approximation, (ii) to a second approximation, (iii) to a third approximation.

What are the values of $1/(0{\cdot}999)^2$ given by each of these three approximations?

14. Use Nos. 11 and 12 to derive formulæ for calculating (i) $1/(1-x)^3$, (ii) $1/(1-x)^4$ approximately, the approximation being carried in each case as far as the term involving x^4.

15. What do the formulæ of Nos. 12 and 14 become when $-x$ is substituted for x?

16. Calculate to a third approximation the values of (i) $1/(0{\cdot}999)^3$, (ii) $1/(1{\cdot}001)^3$, (iii) $1/(1{\cdot}002)^4 - 1/(0{\cdot}998)^4$.

C.

17. Solve the equations :—

(i) $\frac{1}{n} - \frac{2}{n-1} + \frac{1}{n-2} + \frac{n+1}{n(n-1)(n-2)} = 0.$

(ii) $\frac{1}{n-1} - \frac{2}{2n-1} + \frac{1}{3n-1} + \frac{n}{(n-1)(2n-1)(3n-1)} = 0.$

(iii) $\frac{2n}{n-1} + \frac{n+3}{n+1} - \frac{3n+19}{n+5} = 0.$

(iv) $\frac{1}{5n+9} - \frac{3}{5n+11} + \frac{n+7}{25n^2+100n+99} = 0.$

18. Find values of m and n which satisfy the following pairs of relations simultaneously :—

(i) $3m - 2n = 12,\ 9m^2 - 4n^2 = 576.$

(ii) $\frac{1}{5}(m+n) + \frac{1}{3}(m-n) = 2,\ 4m + n = 17.$

(iii) $\frac{5}{m-2n} = \frac{7}{2m-n},\ \frac{3m-2}{7} = \frac{6+m}{5}.$

(iv) $m + 3mn + n + 2 = 0,\ \frac{1}{m} - \frac{1}{n} = 3.$

(v) $\frac{m}{3} - \frac{n}{7} = 2,\ \frac{5}{\frac{1}{n} - \frac{1}{m}} + \frac{2}{\frac{1}{m} + \frac{1}{n}} = 0.$

19. Find values of l, m, and n which satisfy the following relations simultaneously :—

$$\text{(i)}\quad \begin{aligned} &4l - 5m + n + 6 = 0;\\ &7l - 11m + 2n + 12 = 0;\\ &l + m + 3n - 9 = 0. \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} &l + m = 6;\\ &m + n = 28;\\ &n + l = 12. \end{aligned}$$

$$\text{(iii)}\quad \begin{aligned} &3l - 7m + 4n = 1;\\ &5l - 9m + n = -22;\\ &l - 2m + n = 0. \end{aligned}$$

$$\text{(iv)}\quad \begin{aligned} &5l + 2m + 3n = 18;\\ &3l + 7m - n = 5;\\ &l - 2m + n = 6. \end{aligned}$$

$$\text{(v)}\quad \begin{aligned} &\frac{a}{l} + \frac{b}{m} + \frac{c}{n} = 3;\\ &\frac{a}{l} + \frac{b}{m} - \frac{c}{n} = 1;\\ &\frac{2a}{l} - \frac{b}{m} - \frac{c}{n} = 0. \end{aligned}$$

20. Express each of the following as a sum of partial fractions with the simplest possible numerators and denominators :—

$$\text{(i)}\ \frac{2n^2 + 3n - 1}{1 - n + n^2 - n^3}. \qquad \text{(ii)}\ \frac{6n^2 - 17n + 6}{n^3 - 5n^2 + 6n}.$$

$$\text{(iii)}\ \frac{1 + n^2}{(1 - n^2)(1 - n)}.$$

Sketch the graph of (ii) from $n = -3$ to $n = +3$.

EXERCISE XXXIX.

LINEAR FUNCTIONS.

Note.—A line when moved is supposed always to remain parallel to its original direction.

A.

1. Describe in words the positions of the straight lines which correspond to the following relations:—

(i) $y = 1{\cdot}6x$. (ii) $y = 1{\cdot}6x + 4{\cdot}7$.
(iii) $y = 1{\cdot}6x - 8{\cdot}2$. (iv) $y = -2{\cdot}5x + 14{\cdot}3$.
(v) $y = -7{\cdot}8 - 4{\cdot}7x$.

2. Write down the relations which correspond to lines in the following positions:—

(i) Inclined at 31° to the x-axis and raised 4·7 units.
(ii) Inclined at 116° to the x-axis and raised 22 units.
(iii) Inclined at 42° to the x-axis and lowered 12·3 units.
(iv) Inclined at 158° to the x-axis and lowered 11 units.

3. Throw the following equations into the form $y = ax + b$ and give the positions of the corresponding straight lines:—

(i) $7x - 10y - 26 = 0$. (ii) $2{\cdot}8x - 4y + 10 = 0$.
(iii) $3x - 4y + 12 = 0$. (iv) $4x + 3y - 20 = 0$.
(v) $5{\cdot}4x + 3y - 24 = 0$. (vi) $3x - 5{\cdot}4y + 16{\cdot}2 = 0$.

4. Draw on the same sheet the six lines of No. 3. How could you have foreseen (i) that the first pair of lines would be parallel; (ii) that the lines in the second and third pairs would be mutually perpendicular?

5. Show that the lines corresponding to two equations of the form $ax + by + c_1 = 0$ and $bx - ay + c_2 = 0$ must always be at right angles to one another.

6. Find the linear relations that are satisfied by the following pairs of values of the variables. Express them in the standard equational form:—

(i) $(-4, +7)\ (+2, -3)$. (ii) $(+2, +3)\ (-4, +1)$.
(iii) $(-3, +8{\cdot}7)\ (+5, -14{\cdot}5)$. (iv) $(+7, -3{\cdot}2)\ (+7, +4{\cdot}8)$.
(v) $(-14{\cdot}7, -2{\cdot}3)\ (+6{\cdot}4, -2{\cdot}3)$.

7. Calculate (i) the crossing-point of the lines corresponding to the relations in No. 6 (i) and (ii); (ii) the inclination of each of these lines to the x-axis; (iii) the angle between them. Verify by drawing the lines and making the necessary measurements. Decide without drawing a figure what are the positions of the lines corresponding to the relations in No. 6 (iv) and (v). Verify by drawing.

8. Find the value of x for which the functions

$$7x - 3 \text{ and } - 5x + 6$$

have the same value. What is that value? Illustrate your answer by means of the graphs of the functions.

9. Is it possible to find a value of x for which the three functions $-2x - 4$, $\frac{3}{2}x + 17$ and $-4x - 16$ have the same value? If so, what is that value? Illustrate by means of the graphs of the functions.

10. For what value of a do the three functions

$$-3x + 2, + 2x - 3 \text{ and } ax + 4$$

have a common value? What is that value and what value of x produces it? Illustrate by graphs.

B.

11. Draw the graph of tan α from $\alpha = 0°$ to $\alpha = 180°$.

Note.—Take a line through the origin—for example $y = 2x$. Then $y = 2x + 8$ describes the same line raised through 8 units. A figure will show that the line could have been transferred to the same position by moving it 4 units to the left. Thus $y = 2(x + 4)$ implies that the line $y = 2x$ has been shifted 4 units to the left. Similarly $y = 2(x - 4)$ is equivalent to $y = 2x - 8$, and implies that the line has been moved to the right. Since $y = 2x + 8$ can be written as $y - 8 = 2x$ and $y = 2x - 8$ as $y + 8 = 2x$ we have the following rules:—

Substitute $y - 8$ for y and the line is moved *up* 8 units.

Substitute $y + 8$ for y and it is moved *down* 8 units.

Substitute $x - 4$ for x and the line is moved 4 units to the *right*.

Substitute $x + 4$ for x and it is moved 4 units to the *left*.

12. What horizontal movements would produce the same results as the following vertical movements? Write each relation in the forms corresponding to both kinds of movement:—

(i) An upward movement of the line $y = 3x$ through 12 units.
(ii) A downward movement of the line $y = 2{\cdot}4x$ through $7{\cdot}2$ units.
(iii) An upward movement of the line $y = -\frac{1}{4}x$ through 5 units.
(iv) A downward movement of the line $y = -3{\cdot}1x$ through $24{\cdot}8$ units.

13. Give the relations which correspond to the following lines after they have been moved in the manner specified :—

(i) The line $y = 4x - 7$ moved 12 units upwards.
(ii) The same line moved 3 units to the left.
(iii) The line $y = -1{\cdot}4x + 6{\cdot}2$ moved 5 places to the left.
(iv) The same line moved 7 units downwards.
(v) The line $3x - 2y + 6 = 0$ moved 4 places to the right.
(vi) The same line moved 6 places downwards.

14. What movements (*a*) vertically and (*b*) horizontally will effect the changes of position implied by changing the first of each of the following pairs of relations into the second?

(i) $y = 2x + 11$ into $y = 2x - 3$.
(ii) $y = -3x + 8$ into $y = -3x - 7$.
(iii) $3x - 5y + 11 = 0$ into $3x - 5y - 34 = 0$.
(iv) $2x + 7y - 18 = 0$ into $2x + 7y + 10 = 0$.

15. The line $y = 2x$ can be made to pass through the point $(-4, +5)$ by moving it first 4 units to the left and then 5 units upwards. What will now be the corresponding relation?

16. Write down the relations corresponding to the following lines :—

(i) A line through the point $(+3, -7)$ and parallel to $y = 1{\cdot}5x$.
(ii) A line through the point $(-5, -8)$ and parallel to $y = -2{\cdot}3x$.
(iii) A line through the point $(-2, +6)$ inclined at 41° to the x-axis.
(iv) A line through the point $(+8 - 4{\cdot}5)$ inclined at 122° to the x-axis.
(v) A line through the point $(-4, +3)$ perpendicular to the line $y = \frac{3}{4}x$. (See No. 5.)
(vi) A line through the point $(+3{\cdot}2, +1{\cdot}8)$ perpendicular to $2x + 5y - 7 = 0$.

17. The line $y = mx$ when moved so as to pass through the point (p, q) is described by the form $y = mx + c$. Show that $c = -mp + q$.

18. Two lines $y = m_1x$ and $y = m_2x$ when moved so as to pass through (p, q) have as their corresponding relations

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2.$$

Show that $(m_2 - m_1)\,p = c_1 - c_2$.

19. A third line $y = m_3x$ becomes $y = m_3x + c_3$ when moved so that it also passes through the point (p, q). Use the result of No. 18 to prove that when any three lines

$$y = m_1x + c_1,\ y = m_2x + c_2,\ y = m_3x + c_3$$

pass through the same point

$$(m_2 - m_1)/(m_3 - m_2) = (c_1 - c_2)/(c_2 - c_3).$$

20. Apply the test of No. 19 to determine which of the following sets of lines is concurrent:—

(i) $y = 1{\cdot}2x + 8{\cdot}6,\ y = -0{\cdot}2x + 4{\cdot}4,\ y = -3x - 4$.

(ii) $x - 2y - 13 = 0,\ 3x + 2y - 15 = 0,\ 2x - 3y - 23 = 0$.

(iii) $y = 3x - 7,\ y = -2x + 3,\ y = -5x + 9$.

Confirm your conclusions by drawing the lines.

EXERCISE XL.

DIRECTED TRIGONOMETRICAL RATIOS.

A.

1. Draw the graph of $\sin \alpha$ from $\alpha = 0°$ to $\alpha = 180°$.

2. Directly below the former graph and with the same scales draw the graph of $\cos \alpha$ for the same values of α.

3. By comparing the graphs show that $\sin (\alpha + 90°) = \cos \alpha$, α being less than $90°$. Complete the identity

$$\cos (\alpha + 90°) = \ldots,$$

α being less than $90°$.

4. Find equivalents for $\sin (\alpha - 90°)$ and $\cos (\alpha - 90°)$ when $180° > \alpha > 90°$.

5. From the results of Nos. 3 and 4 find equivalents for $\tan (\alpha + 90°)$, $(\alpha < 90°)$, and $\tan (\alpha - 90°)$, $(180° > \alpha > 90°)$. Do the results agree with the graph of Ex. XXXIX, No. 11?

Note.—The reciprocal of the tangent of an angle is called the **cotangent** of that angle. In symbols,

$$\cot \alpha = 1/\tan \alpha = \cos \alpha/\sin \alpha.$$

6. Show by a figure that $\tan (90° - \alpha) = \cot \alpha$ and that $\cot (90° - \alpha) = \tan \alpha$, when $\alpha < 90°$.

7. Find similar equivalences for $\tan (\alpha - 90°)$ and $\cot (\alpha - 90°)$ when $90° < \alpha < 180°$.

8. Sketch roughly, for comparison, the graphs of $\tan \alpha$ and $\cot \alpha$ from $\alpha = 0°$ to $\alpha = 180°$. Find equivalences for $\tan (\alpha + 90°)$, $\cot (\alpha + 90°)$. Do the results agree with those of No. 5?

9. An officer in a battery on an island determines by his range-finder that two ships are respectively 1200 and 1800 yards distant. He also observes that the angle between them is 53°. How far are they from one another?

10. Half an hour later the officer observes that the ships are at the same distances as before from his battery, but that

the angle between them is 113°. What is now the distance between the ships?

11. The road from a certain village, X, runs practically straight for 2 miles. It then changes its direction by 32°. The village of Y lies 1·3 miles from the turning. What is the direct distance from X to Y? What angle does the line joining the villages make with the road out of X?

12. The difference in direction between two vectors is δ and their lengths are respectively a and b. Show that the length of their resultant is $\sqrt{(a^2 + b^2 + 2ab \cos \delta)}$. If the resultant makes an angle of β with the vector a, show that

$$\sin \beta = b \sin \delta / \sqrt{(a^2 + b^2 + 2ab \cos \delta)}.$$

B.

13. The sides of a triangle are proportional to the numbers 9, 12, 20. Calculate the angles.

14. Three church spires mark a triangle whose sides are 2 miles, 3 miles, and 4 miles long. Calculate the angle subtended by each pair of spires as seen from the remaining one.

15. The base of a triangle is 120 yards long; and the angles at the base are respectively 27° and 122°. Calculate the third angle and the lengths of the other two sides.

16. Calculate the area of the triangle of No. 15.

17. The length of the base of a triangle is c and the base angles are respectively α and β. Show that the area is given by the formula

$$A = \frac{c^2 \sin \alpha \sin \beta}{2 \sin (\alpha + \beta)}.$$

18. Calculate the area of a triangle in which $c = 14°$, $\alpha = 71°$, $\beta = 43°$.

19. Show that the altitude of an isosceles triangle is $\frac{1}{2}c \tan \alpha$, and that its area is $\frac{1}{4}c^2 \tan \alpha$, α being the base angle.

20. The magnitude of the angle A in a triangle is α. Show that

$$\sin^2 \alpha = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2.$$

Hence show that

$$\sin \alpha = \frac{1}{2bc} \cdot \sqrt{\{(a + b + c)(a + b - c)(b + c - a)(c + a - b)\}}.$$

21. Use the foregoing formula to calculate the angles of a triangle whose sides are respectively 3, 4, and 5 inches long.

22. Show that if $2s$ is substituted for $(a + b + c)$ the equivalence of No. 20 becomes

$$\sin \alpha = \frac{2}{bc}\sqrt{\{s(s - a)(s - b)(s - c)\}}.$$

Hence show that the area of the triangle is given by the formula

$$A = \sqrt{\{s(s - a)(s - b)(s - c)\}}.$$

23. The sides of a triangle are respectively 60, 40, and 80 yards long. Calculate its area.

24. The sides AB, BC, CD, DA, of an irregular four-sided field are, in order, 400 feet, 350 feet, 270 feet, and 320 feet long. The diagonal AC is 520 feet long. Calculate the area of the field and the size of its angles.

EXERCISE XLI.

SURVEYING PROBLEMS.

A.

Note.—Surveyors in making a map of a district begin by fixing the relative positions of prominent points (e.g. a flag on a church tower, a solitary tree on a hill) by means of a series of triangles. Fig. 40 illustrates such a **triangulation.**

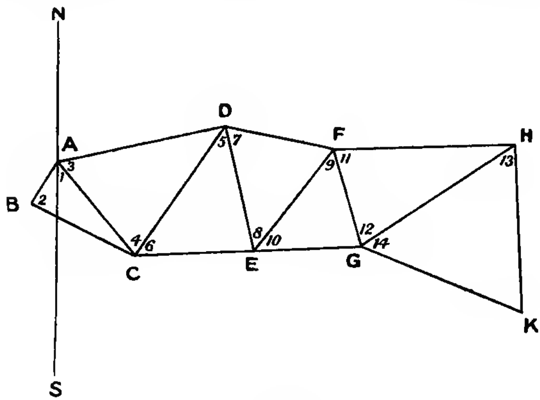


FIG. 40.

The length of a **base** AB is determined (on level ground) with extreme care, and the angles 1, 2, 3, . . . 14 are measured with a **theodolite.** By the theodolite the angle of elevation or depression of each station as viewed from the preceding station is also determined. From these measurements the lengths of the sides of the triangles and the heights or depths of each station above or below A can be calculated. Finally the co-ordinates of the stations B, C, D, etc., are calculated with reference to the north and south and east and west lines through A.

In No. 1 the calculation is to be divided among the class. Group I are to calculate AC and BC; Group II are to assume AC $= b$ and to calculate CD and AD; Group III are to assume CD $= c$ and to calculate DE and DF; and so on with the other groups. When Group I have calculated AC, Group II are to substitute its value in their expression for CD; Group III are to substitute Group II's result for CD in their expression for DE, and so on. While a group are waiting for the result of the previous group they may solve Nos. 2, 3, and 4.

1. The length of the base AB is exactly 1 mile. The angles are as follows:—

Number:	1	2	3	4	5	6	7	8
Angle:	72°	83°	63°	74°	48°	52°	67°	51°
Number:	9	10	11	12	13	14		
Angle:	56°	49°	76°	72°	60°	55°		

Calculate the lengths necessary for a careful drawing of the triangles by graduated ruler and compasses.

2. The stations C and D are on the crest of a ridge. At A the angle of elevation of C is 4°. Calculate the height of C above A in feet.

3. The angle of elevation of D from C is 2°. Calculate the height of D above A.

4. The angle of elevation of D from A is 4° 42′. Calculate the height of D above A. Does the calculation agree with the result of No. 3?

5. E is on the eastern side of the ridge CD. The angle of depression of E from C is 2° 36′ and from D 4° 56′. Calculate from both observations the height of E above A.

6. The angle BAS is 32°. Calculate the bearings of (i) C from A, (ii) D from C, (iii) E from D, . . . (vii) K from H. Estimate all the bearings in degrees from the north round by the east. (Thus a line bearing 30° E. of S. is to be given as bearing 150° from the north.)

Note.—In No. 7 the work is to be divided as in No. 1. Group I are to find the distances of C from the NS and EW lines through A; Group II the distances of D from the NS and EW lines through C, etc.; Group VII the distances of K from the NS and EW lines through H. The results are to be expressed in directed numbers.

7. Calculate the co-ordinates of each of the points C, D, E, G, H, K, with respect to NS and EW lines through A, C, D, E, G, H respectively. From the results determine the

co-ordinates of all the points with respect to NS and EW lines through A.

8. Make a map in which the points A, B, . . . K are inserted in their correct positions.

B.

Note.—An alternative, less elaborate method of surveying is by making a **traverse.** The details of the country within a triangulation are often fixed in this way. The method consists in determining the lengths and the bearings of the lines leading from each of the stations A, B, C, D, etc. (fig. 41) to

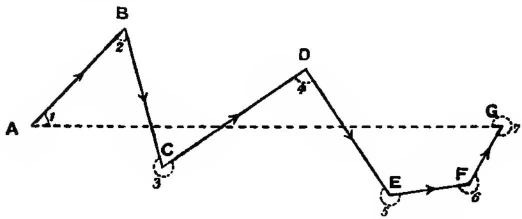


FIG. 41.

the next. The lengths are measured with a chain or a steel tape, the bearings with a **prismatic compass** for rapid work or a theodolite in careful surveying. In the former case the angle between each of the directions AB, BC, CD, . . . GA may be measured, and also the bearing of AB. From these measurements the bearings of the other lines are calculated. In the second case the angles 1, 2, 3, . . . are accurately determined. The map is made, either by drawing vectors, or, in more accurate work, by calculating the co-ordinates of the stations with respect to axes through A. If the lengths and angles have been correctly measured the traverse ought to **close**—that is, the last line, GA, should make with the others a closed polygon.

Bearings are always taken continuously from the north round by the east. Thus a point 10° E. of S. is taken to have a bearing of 170°, a point 10° W. of S. one of 190°, a point 40° W. of N. one of 320°. In order to calculate easily the co-ordinates of the points in the traverse every angle from 0° to 360° is supposed to have its own sine and cosine.

9. Draw a line NOS to represent the meridian. From O

draw a vector of length r with a bearing 220°. Draw the component western or southern vectors or resolved parts OQ, QP. By consideration of these determine what sine, cosine, and tangent you will assign to 220°.

10. Answer the same questions with reference to a vector whose bearing is 310°.

11. State the equivalences by which you will obtain sin α, cos α, and tan α from the tables on pp. 107 and 111 (i) when $180° > \alpha > 90°$, (ii) $270° > \alpha > 180°$, (iii) $360° > \alpha > 270°$.

12. The lengths and bearings of the lines in the traverse shown in fig. 42 are given below. Each member of the class

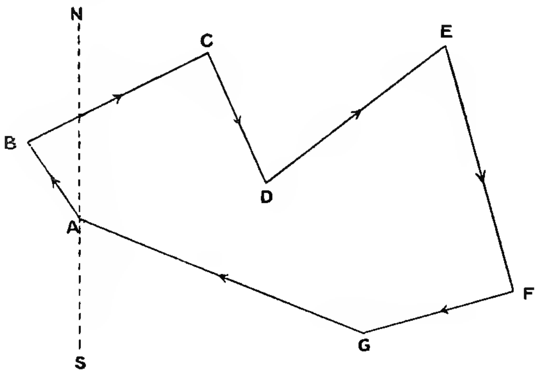


FIG. 42.

is to calculate the components of two of the vectors. From the combined results each one is to calculate the co-ordinates of B, C, . . . G and A with reference to the NS and EW lines through A.

	AB	BC	CD	DE	EF	FG	GA
Distances	250	510	384	608	691	422	710 yards.
Bearings	326°	64°	156°	52°	165°	254°	297°

13. How far is the traverse from being perfectly closed? [Calculate the co-ordinates of A from GA.]

14. In a traverse connecting seven stations, A to G, angles corresponding to the angles 1, 2, 3, . . . 7 of fig. 41 are measured with a theodolite and are as follows :—

1	2	3	4	5	6	7
47°	58°	289°	89°	242°	233°	302°

Calculate the *difference* of bearing between each of the lines AB, BC, . . . FG, and AG. (That is, if G were due north of A, what would be the bearings of the lines AB, BC, etc.?)

15. The lengths of the lines in the traverse of No. 14 are as follows. Calculate the co-ordinates of the points B, C, D, E, F, G, with reference to axes through A respectively parallel and perpendicular to AG.

AB	BC	CD	DE	EF	FG
362	389	470	409	212	180 yards.

16. Sketch roughly the graphs of sin α, cos α, and tan α from $\alpha = 0°$ to $\alpha = 360°$. Do the identities of Ex. XL, Nos. 3 and 8 hold good for all angles between 0° and 270°? Do those of Nos. 4 and 7 hold good for all angles between 90° and 360°?

C.

Note.—Except in surveying, an angle is reckoned positive if measured in the anti-clockwise, and negative if measured in the clockwise, direction.

17. Take a line of length OP making $-48°$ with the x-axis. From O draw PM, PN perpendicular to OX and OY. What values must be assigned to $\cos(-48°)$ and $\sin(-48°)$ so that the rules $PM = l \cos \alpha$ and $PN = l \sin \alpha$ may be observed?

18. Determine by the same principle the sines and cosines of: (i) $-100°$, (ii) $-200°$, (iii) $-300°$.

19. Find the tangents of: (i) $-23°$, (ii) $-135°$, (iii) $-250°$, (iv) $-340°$.

20. Make a table showing the signs of (i) positive, and (ii) negative angles in each of the four quadrants.

EXERCISE XLII.

HYPERBOLIC AND PARABOLIC FUNCTIONS.

A.

1. Draw on tracing paper, in accordance with instructions, the rectangular hyperbola $xy = k$ or $y = k/x$, including its asymptotes. Either choose your own value for k (selecting some positive number) or work with the one assigned to you. In Nos. 2 and 3 you are supposed to start with the asymptotes of your curve coincident with axes of x and y drawn on a sheet of squared paper lying beneath the tracing paper.

2. Carry out the following movements with your curve and give in each case the formula which describes it in its new position :—

Move the curve $y = k/x$ (i) 6 units upwards ; (ii) 14 units to the left ; (iii) 17 units to the left ; (iv) 15 units downwards ; (v) 23 units downwards and 10 units to the right ; (vi) 16 units to the left and 9 units upwards ; (vii) so that the crossing-point of the asymptotes is at the point $(-8, +12)$; (viii) so that it is at the point $(+18, -15)$.

3. Move your curve successively into the positions corresponding to the following relations. Describe the movements in words :—

(i) $\mathbf{y = k/(x - 9)}$. (ii) $\mathbf{y - 8 = k/x}$.
(iii) $\mathbf{y + 11 = k/(x - 13)}$. (iv) $\mathbf{y = k/(x + 16) + 9}$.

4. Place your tracing paper upon the squared paper so that the curve on it corresponds successively to the following relations :—

(i) $\mathbf{y = -k/x}$. (ii) $\mathbf{y + 13 = -k/x}$.
(iii) $\mathbf{y = 22 - k/(x + 7)}$. (iv) $\mathbf{y = 22 + k/(x + 7)}$.
(v) $\mathbf{y + 11 = -k/(x - 23)}$. (vi) $\mathbf{y + 11 = k/(x - 23)}$.

5. Throw each of the following equations into the form :—

$$y \pm b = \frac{k}{x \pm a}$$

(i) $xy + 3x - 12 = 0$. (ii) $xy + 3y - 12 = 0$.
(iii) $xy - 5x + 7 = 0$. (iv) $3xy + 4y + 15 = 0$.
(v) $xy + 3x + 4y + 6 = 0$. (vi) $xy + 3x + 5y = 0$.
(vii) $2xy + 7x + 3y - 8 = 0$. (viii) $3xy - 2x - 6y = 0$.

6. In the case of each of the curves of No. 5 state: (*a*) what two rectangular movements would bring its asymptotes into coincidence with the axes of x and y; (*b*) what relation would correspond to the curve in this position.

7. Find the positions of the two vertices of No. 5, (i), (iv), (viii). [Find their positions when the asymptotes are coincident with the axes of x and y. Then suppose the curve to be restored to the position in which it was given.]

8. Find the hyperbolic relations which are satisfied by the following sets of values of x and y:—

(i) $(+8, +9)$, $(-5, -4)$, $(+10, +6)$.
(ii) $(+3, -43)$, $(+22, -5)$, $(-6, +2)$.
(iii) $(0, +12)$, $(-5, +14)$, $(-20, +8)$.

Illustrate your solution by graphs.

Note.—Assume $y + b = k/(x + a)$ and determine the values of the constants. To draw the graphs easily, first fix the position of the asymptotes; then regard them as if they were axes of x and y and plot the curve $y = k/x$.

B.

9. Draw on tracing paper as in No. 1 the parabola

$$y = kx^2$$

k being a positive number chosen by you or assigned to you. Let your drawing include the axis and the tangent at the vertex.

10. Move your curve successively from the position in which its axis coincides with the y-axis and the tangent at the vertex coincides with the x-axis into the positions which correspond to the following relations. State in each case whether the parabola is "head up" or "head down" and give the co-ordinates of the vertex:—

(i) $y = kx^2 + 7$. (ii) $y = -kx^2$.
(iii) $y = -kx^2 + 7$. (iv) $y + 12 = kx^2$.
(v) $y - 20 = -kx^2$. (vi) $y = k(x + 14)^2$.
(vii) $y = -k(x - 14)^2$. (viii) $y = k(x + 16)^2 - 21$.
(ix) $y + 21 = -k(x - 13)^2$. (x) $y - 15{\cdot}5 = -k(x + 7{\cdot}8)^2$.

11. Give the relations corresponding to your parabola when it is held in the following positions:—

(i) Head down with the vertex at $(-6, +4)$.
(ii) Head up with the vertex at $(-9, -11)$.
(iii) Head up with the vertex at $(+14, -7)$.
(iv) Head down with the vertex at $(0, -17)$.

12. In the case of each of the following functions of x state: (*a*) whether it has an upper or a lower turning value; (*b*) what that value is; (*c*) what value of x gives it the turning value:—

(i) $y = 6(x - 7)^2 + 4$. (ii) $y = -3(x + 5)^2 + 7$.
(iii) $y = -3{\cdot}6(x - 7{\cdot}2)^2 - 1{\cdot}8$. (iv) $y = -7x^2 - 9{\cdot}3$.
(v) $y = 4(x + 2{\cdot}3)^2$. (vi) $y = (x + 7{\cdot}8)^2 - 21{\cdot}3$.

13. Express each of the following relations in the form

$$y = \pm a(x \pm b)^2 \pm c.$$

Describe, as in No. 10, the position of each of the corresponding parabolas:—

(i) $y = x^2 - 6x + 3$. (ii) $y = x^2 + 10x - 2$.
(iii) $y = -x^2 + 12x + 7$. (iv) $y = 3x^2 - 12x + 5$.
(v) $y = -7x^2 + 28x - 11$. (vi) $y = x^2 + 5x - 1$.
(vii) $y = -2x^2 + 14x + 3$. (viii) $y = 6x^2 - 14x$.
(ix) $y = -4x^2 + 13x$. (x) $y = 2{\cdot}3x^2 + 11{\cdot}5x - 7{\cdot}2$.

14. Answer the questions of No. 12 with regard to each of the functions of x in No. 13.

Note.—Consider the relation $y = 4x^2 - 8x - 5$. It can be transformed as follows:—

$$\begin{aligned} y &= 4x^2 - 8x - 5 \\ &= 4(x^2 - 2x + 1) - 9 \\ &= 4(x - 1)^2 - 3^2 \\ &= \{2(x - 1) + 3\}\{2(x - 1) - 3\} \\ &= (2x + 1)(2x - 5) \end{aligned}$$

$$\begin{aligned} \text{Similarly } y &= 3x^2 - 30x + 48 \\ &= 3(x^2 - 10x + 16) \\ &= 3\{(x^2 - 10x + 25) - 9\} \\ &= 3\{(x - 5)^2 - 3^2\} \\ &= 3(x - 8)(x - 2) \end{aligned}$$

This process is described by saying that the **parabolic** (or quadratic) **function** $4x^2 - 8x - 5$ has been expressed as the **product of two linear functions** of x, namely

$$2x + 1 \text{ and } 2x - 5.$$

15. Where possible express each of the following parabolic functions of x as a product of two linear functions. How could you tell from the graphs of the functions in which

cases the transformation is possible and impossible respectively?

(i) $y = x^2 + 6x - 7$. (ii) $y = x^2 + 6x + 11$.
(iii) $y = 4x^2 - 16x - 34$. (iv) $y = 4x^2 - 16x + 18$.
(v) $y = 3x^2 + 36x + 27$. (vi) $y = -3x^2 + 36x - 27$.
(vii) $y = 5x^2 + 17x - 12$. (viii) $y = 2x^2 - 6{\cdot}4x - 7{\cdot}6$.

16. How could you tell from the graphs of the functions of No. 15 (*a*) which of them is capable of having the value 0; (*b*) what values of x make the value of the function 0?

Note.—The easiest way to calculate where the parabola $y = 4x^2 - 8x - 5$ crosses the x-axis is to express the quadratic function $4x^2 - 8x - 5$ as a product of two linear functions. We then have:—

$$4x^2 - 8x - 5 = (2x + 1)(2x - 5) = 0$$

Therefore either $2x + 1 = 0$ or $2x - 5 = 0$;
that is, either $x = -1/2$ or $x = +5/2$.

17. Find the values of x for which (where it is possible) the various functions of No. 15 have zero value.

18. The vertex of a parabola is at the point $(+4, +7)$ and passes through the point $(-5, -20)$. Find the formula.

19. Another parabola has its vertex at the same point and passes through $(+13, +34)$. Find the corresponding relation.

20. Indicate by a sketch the positions of the parabolas of Nos. 18 and 19.

21. When $x = -6$ a certain parabolic function of x has a turning value of $+72$. When $x = 0$ its value is zero. Express the function in the form $px^2 + qx + r$. What kind of turning value has it?

22. Find the parabola which has the line $x = -10$ as its axis, passes through the point $(+4, +40)$ and crosses the x-axis at $x = -4$. Where is its vertex?

23. The values of a parabolic function of x are the same for all values of the variable which are at an equal distance above and below $+6$. Its value is zero when $x = -1$ and -26 when $x = +12$. Find the function and give its turning value.

C.

Note.—The best way to determine whether a given curve is a parabola is to measure a number of ordinates, equidistant,

but otherwise taken at random. As you have seen, if the curve is parabolic the second differences of the ordinates will be constant.

When you know that the curve is a parabola the easiest way to find its formula is to note the co-ordinates of its

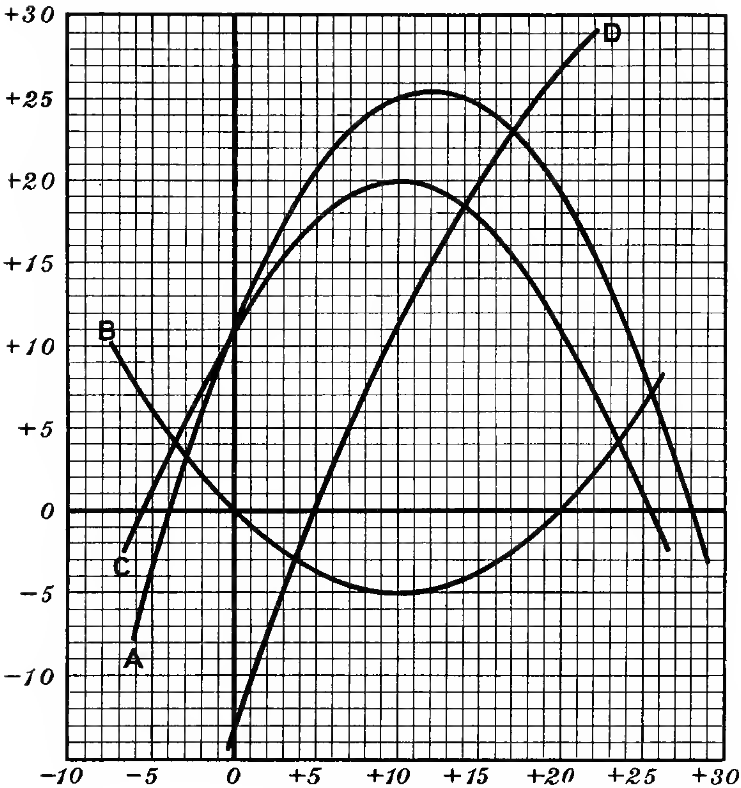


FIG. 43.

vertex and of some other point—for example where it crosses the x-axis or the y-axis. The method of Nos. 18, 19 should then be applied. If the position of the vertex is not shown find the co-ordinates of any three convenient points and use the values of x and y to determine the constants in

$$y = px^2 + qx + r.$$

24. Determine whether the curve A in fig. 43 is a parabola. If it is find its formula.

25. Repeat with curve B.

26. Repeat with curve C.

27. Determine whether curve D is a portion of a parabola. If so find the formula and state the position of the vertex.

28. Roll an oiled ball diagonally up a sloping drawing-board as instructed. Show that the trace is a parabola. Find its formula, taking a vertical and a horizontal edge of your paper as axes. What is the connexion between the vertical and horizontal distances of a projectile from the point of projection?

29. Arrange that a fine jet of water may be projected horizontally with constant pressure and may fall into a sink some distance away from, and below, the nozzle of the jet-tube. Find whether the middle of the stream is a parabola.

30. Repeat, inclining the jet-tube upwards. Deal separately with the inmost and outmost portions of the stream. If they are parabolas calculate the positions of their turning-points. Also determine by calculation whether they pass through the mouth of the nozzle.

31. The following table gives the values of a certain function for given values of x. Find whether the function is parabolic. If so determine its precise form and its turning value :—

x:	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
y:	$+9{\cdot}8$	$+7{\cdot}9$	$+6{\cdot}2$	$+4{\cdot}7$	$+3{\cdot}4$	$+2{\cdot}3$	$+1{\cdot}4$	$+0{\cdot}7$	$+0{\cdot}2$

32. Repeat the investigation upon the following data :—

x:	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
y:	$+13$	$+6$	$+2$	$+0$	-1	-2	-4	-8	-15

EXERCISE XLIII.

QUADRATIC EQUATIONS.

A.

1. Calculate to two decimal places the abscissæ of the points on the following parabolas where the ordinates have the specified values:—

(i) The parabola $y = 3(x - 2)^2 - 5$ where $y = +7$, $y = -2$, $y = 0$.
(ii) The parabola $y = \quad 4(x + 3)^2 + 20$ where $y = -80$, $y = +12$, $y = 20$.
(iii) The parabola $y = 2x^2 - 7x - 3$ where $y = +\frac{1}{16}$, $y = -5\frac{15}{16}$.
(iv) The parabola $y = 3 - 7x - 7x^2$ where $y = 0$, $y = +3$.

2. Find the values of x for which

(i) The function $x^2 - 3x - 10$ has the value $+3{\cdot}75$.
(ii) The function $x^2 + 0{\cdot}9x + 10$ has the value $-3{\cdot}7$.
(iii) The function $9x^2 + 3x - 8$ has the value -6.
(iv) The function $8{\cdot}2 - 5{\cdot}1x - 3x^2$ has the value $+10$.

3. Find by direct factorization the roots of the following quadratic equations:—

(i) $x^2 - 4x + 3 = 0$.	(ii) $x^2 + 4x - 5 = 0$.
(iii) $x^2 + 12x + 36 = 0$.	(iv) $x^2 \quad 13x + 40 = 0$.
(v) $x^2 - 3x - 40 = 0$.	(vi) $x^2 - 15x + 36 = 0$.
(vii) $x^2 - 7x = 0$.	(viii) $x^2 + 5{\cdot}3x = 0$.
(ix) $2x^2 - 7x + 3 = 0$.	(x) $2x^2 + 2x - 3 = 0$.
(xi) $6x^2 - 13x + 6 = 0$.	(xii) $8x^2 + 26x - 7 = 0$.
(xiii) $1 - 8x + 15x^2 = 0$.	(xiv) $2x^2 + 7x + 6 = 0$.
(xv) $6x^2 - 13x - 5 = 0$.	(xvi) $4x^2 - 20x + 25 = 0$.

Note.—How can you form a quadratic equation whose roots shall be $+2$ and $+3$? -2 and -3? -2 and $+3$? α and β? What is the relation between the roots and (i) the coefficient of x, (ii) the constant term in the equation?

4. Quadratic equations are to be formed having the following pairs of roots. Set down in four parallel columns the roots, the coefficients of x in the equations, the constant terms, the completed equations:—

(i) $-3, +5$. (ii) $+7, -8$.
(iii) $+4, +3$. (iv) $-10, -10$.
(v) $-10, +10$. (vi) $+5{\cdot}2, -2{\cdot}5$.
(vii) $+0{\cdot}3, +2{\cdot}1$. (viii) $+3 \pm 5$.
(ix) $+2{\cdot}3 \pm 1{\cdot}2$. (x) $+5+\sqrt{3}, +5-\sqrt{3}$.
(xi) $-1 \pm \sqrt{2}$. (xii) $+1{\cdot}3 \pm \sqrt{5{\cdot}4}$.
(xiii) $a \pm b$. (xiv) $pa^2 \pm qb^2$.
(xv) $+\sqrt{2} \pm 3$. (xvi) $-\sqrt{5} \pm \sqrt{7}$.

5. The roots of a quadratic equation are of the form $a \pm \sqrt{b}$. Show that the coefficient of x and the constant term are both rational. Is this the case if the roots are of the form $\sqrt{a} \pm b$?

6. The roots of a quadratic equation are of the form $\sqrt{a} \pm \sqrt{b}$. Show that the constant term is rational but that the coefficient of x is not. Is this the case if the roots are of the form $\sqrt{a} \pm b$?

7. State the sum of the roots and the product of the roots of each of the following quadratic equations:—

(i) $3x^2 + 5x - 1 = 0$. (ii) $2x^2 - 7{\cdot}3x + 4{\cdot}7 = 0$.
(iii) $3{\cdot}2x^2 - 8{\cdot}4x - 5 = 0$. (iv) $px^2 + qx + v = 0$.
(v) $(a + b)x^2 - (a - b)x - ab = 0$.

8. Write down, in a form clear of fractions, the quadratic equation whose roots are:—

(i) $-\frac{1}{3}, +\frac{3}{4}$. (ii) $+\frac{3}{7}, +\frac{2}{5}$. (iii) $+\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{5}}$.
(iv) $-\frac{1}{a}, -\frac{1}{b}$. (v) $+\frac{a}{b^2}, +\frac{b}{a^2}$. (vi) $\frac{p-q}{p+q}, \frac{p+q}{p-q}$.

9. Use the results of No. 6 to solve the following equations:—

(i) $x^2 - 2\sqrt{3}x + 2 = 0$. (ii) $x^2 + 2\sqrt{5}x - 11 = 0$.
(iii) $x^2 - 2\sqrt{13}x + 5 = 0$. (iv) $x^2 + 2\sqrt{15} - 9 = 0$.
(v) $x^2 - 2\sqrt{a}x + b = 0$. (vi) $x^2 + 2\sqrt{(a+b)}x + (a-b) = 0$.
(vii) $x^2 - \sqrt{7}x - \frac{1}{2} = 0$. (viii) $x^2 - \sqrt{14}x + 3 = 0$.

10. If you are given one root of a quadratic equation how can you calculate the other (i) from the coefficient of x, (ii) from the constant term? Use one of these methods alternately to find the second roots in the following instances. Use the other method to check the result in each case.

(i) $x^2 + 7{\cdot}7x + 14{\cdot}4 = 0$; given root, $-3{\cdot}2$.
(ii) $x^2 - 7{\cdot}4x - 1{\cdot}52 = 0$; given root, $-0{\cdot}2$.
(iii) $x^2 + 2{\cdot}4x - 46{\cdot}11 = 0$; given root, $+5{\cdot}3$.
(iv) $x^2 - 7{\cdot}7x + 14{\cdot}62 = 0$; given root, $+3{\cdot}4$.
(v) $33x^2 - 7x - 10 = 0$; given root, $+\frac{2}{3}$.
(vi) $63x^2 + 22x - 21 = 0$; given root, $+\frac{3}{7}$.

B.

11. Find which of the following parabolas crosses the x-axis :—

(i) $y = x^2 - 4x - 3$. (ii) $y = x^2 - 4x + 5$.
(iii) $y = 3x^2 + 4x + 2$. (iv) $y = -2x^2 + 3x + 7$.
(v) $y = -2x^2 - 3x - 5$. (vi) $y = 3x^2 - 12x + 12$.

12. Describe in symbols the steps by which the problems of No. 11 are solved. Use p, q, and r as symbols for the coefficient of x^2, the coefficient of x and the constant term respectively. Hence show that (whether the parabola is "head down" or "head up") it will cross the x-axis provided that $q^2 - 4pr$ is positive, and that it will touch the x-axis if

$$q^2 = 4pr.$$

13. Write down the condition (i) that the function

$$ax^2 + bx + c$$

may be capable of having zero value; (ii) that it may be represented as the product of two linear functions; (iii) that it may be a perfect square; (iv) that the quadratic equation

$$ax^2 + bx + c = 0$$

may have roots; (v) that the roots may be identical.

14. Apply the tests of No. 13 to classify the following equations into those which have (*a*) identical roots, (*b*) unequal roots, (*c*) no roots :—

(i) $4x^2 - 12x + 9 = 0$. (ii) $3x^2 - 7x + 4 = 0$.
(iii) $x^2 + px + p^2 = 0$. (iv) $5x^2 + 6x - 8 = 0$.
(v) $0{\cdot}16x^2 - 0{\cdot}8x + 1 = 0$. (vi) $3x^2 - 7x + 4 = 0$.

15. Determine which of the following equations have roots. Determine the roots, where they exist, by expressing the left-hand side of the equation as the difference between two squares and then factorizing it. Leave the roots, if they are not rational, in the form of a pair of conjugate surds.

(i) $x^2 - 11x + 12 = 0$. (ii) $x^2 - 11x - 12 = 0$.
(iii) $x^2 + 4x - 7 = 0$. (iv) $2x^2 - 3x + 5 = 0$.
(v) $4x^2 + 4x - 5 = 0$. (vi) $2x^2 - 5x + 32 = 0$.
(vii) $2x^2 + 13x - 3 = 0$. (viii) $x^2 - 2\sqrt{11}x + 20 = 0$.
(ix) $x^2 - \sqrt{7}x + \sqrt{15}$. (x) $\sqrt{3}x^2 - \sqrt{18}x - 1 = 0$.

16. The roots of a quadratic equation are α and β. Write down the equation whose roots are $m\alpha$ and $m\beta$.

17. Give the equations whose roots are respectively :—

(i) Four times those of the equation $x^2 - 2x - 3 = 0$.
(ii) Ten times those of the equation $x^2 - 1\cdot3x - 7\cdot14 = 0$.
(iii) One-fifth of those of the equation $x^2 + 5x - 150 = 0$.

Verify the transformation in each case.

18. Use the result of No. 16 to derive from each of the following equations another in which (*a*) the coefficient of x^2 is unity, (*b*) the coefficient of x and the constant term are both whole numbers. Find the roots of the transformed equations and deduce from them the roots of the original equations. Verify your results :—

(i) $7x^2 + 4x - 3 = 0$. (ii) $14x^2 - 19x + 6 = 0$.
(iii) $17x^2 - 11x - 6 = 0$. (iv) $19x^2 + 9x - 10 = 0$.

19. Show how to calculate the values of a variable x defined by relations of the following types :—

(i) $ax^2 - (a - b)x - b = 0$.
(ii) $x^2 - p(x - q) + q^2 = 0$.
(iii) $pqx^2 + (p - q)x = 1$.
(iv) $abx^2 + (a^2 + b^2)x + ab = 0$.
(v) $2(a - b)(x^2 + b) = (a + b)^2x$.
(vi) $x^2 - 2ax + a^2 = b^2$.
(vii) $(a - b)^2x^2 - 2(a^2 - b^2)x + (a + b)^2 = b^2$.
(viii) $(a - b)^2x^2 + (a^3 - b^3)x + ab(a^2 + b^2) = 0$.

20. Find, where possible, values of x for which the following pairs of functions have the same value :—

(i) $\frac{1}{2}x^2 - 3x + 4\cdot2$ and $\frac{1}{2}x^2 + 5x - 5\cdot8$.
(ii) $x^2 + 2x + 7$ and $4x - 3$.
(iii) $2x^2 - 3x + 10$ and $-2x + 7$.
(iv) $x^2 + 2x + 7$ and $2x^2 - 3x + 10$.
(v) $x^2 + 2x + 7$ and $\frac{1}{2}x^2 + x + 3$.
(vi) $x^2 + 2x + 7$ and $-\frac{1}{2}x^2 - x - 3$.
(vii) $2x^2 - 3x + 10$ and $\frac{1}{2}x^2 + x + 3$.
(viii) $2x^2 - 3x + 10$ and $-\frac{1}{2}x^2 - x - 3$.

Explain the result in (i) by a sketch. Verify the results in (ii)-(viii) by drawing graphs of the functions all upon the same paper.

EXERCISE XLIV.

FURTHER EQUATIONS.

A.

1. Write the equation whose roots are $2a$ times those of $ax^2 + bx + c = 0$. Hence show that the roots of the latter are $\{-b \pm \sqrt{(b^2 - 4ac)}\}/2a$. Does this result agree with those of Ex. XLIII, No. 12?

Note.—The roots of a quadratic equation may be calculated directly by means of the formula of No. 1.

2. Which of the following equivalences is possible and which impossible? Determine in the former cases the values of x which satisfy the relation.

(i) $x + \dfrac{1}{x} = 2x + \dfrac{1}{2x}$. (ii) $x + \dfrac{3}{x-1} = 6$.

(iii) $x + \dfrac{3}{x-1} = 4$. (iv) $2(x-1) + \dfrac{1}{x+3} = 1$.

(v) $\dfrac{1}{x-2} - \dfrac{1}{x+2} = 3$. (vi) $\dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x+2}$.

(vii) $\dfrac{3}{x} - \dfrac{2}{x+1} = \dfrac{1}{x+2}$. (viii) $\dfrac{1}{x} - \dfrac{2}{x+1} = \dfrac{3}{x+2}$.

3. Explain the results of No. 2 (ii) and (iii), by drawing the graph of $y = x + 3/(x-1)$ from $x = 0$ to $x = +6$. What turning value has the function $x + 3/(x-1)$ in this region?

4. Replace the improper fractions in the following relations by their equivalent integral expressions and complementary fractions. Simplify the reduced relations and find the values of x (if there are any) which satisfy them.

(i) $\dfrac{2x-1}{x} + \dfrac{4x}{2x+1} = 1$.

(ii) $\dfrac{2x-3}{x} - \dfrac{2x+7}{x+1} = \dfrac{37}{20}$

(iii) $\dfrac{x^2 - 2x + 3}{x} - 6 = \dfrac{2x^2 - 3x + 1}{2x + 1}$.

(iv) $\dfrac{12x + 11}{4x + 3} - \dfrac{2x}{2x - 1} = \dfrac{2x + 5}{x + 2}$.

5. Explain the result of No. 4 (i) by plotting the graph of $y = 2/(2x + 1) - 1/x$ from $x = -4$ to $x = +4$.

B.

6. Form equations of the lowest possible degree with the following sets of roots :—

(i) $-1, +1, +2$.
(ii) $+1, +2, +3$.
(iii) $-2, 0, +2$.
(iv) $-4, -2, +3$.
(v) $-2, -1, +1, +2$.
(vi) $-3, -2, 0, +1$.
(vii) $-3, -1, +2, +4$.
(viii) $-3, 0, 0$.
(ix) $-6, 0, 0, +5$.
(x) $-2, -1, 0, 0, +3$.

Verify your result in (ii) and (v).

7. Find the roots of the following equations by factorization :—

(i) $x^4 - 13x^2 + 36 = 0$.
(ii) $x - 4\sqrt{x} + 3 = 0$.
(iii) $x^2 - 17x + 16 = 0$.
(iv) $x + 12\sqrt{x} + 35 = 0$.
(v) $x^3 - 3x^2 - x + 3 = 0$.
(vi) $x^3 + 2x^2 - 4x - 8 = 0$.
(vii) $x^3 - 7x^2 + 12x = 0$.
(viii) $x^4 + 3x^3 - 40x^2 = 0$.
(ix) $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$.
(x) $x^3 - 6x^2 + 12x - 8 = 0$.
(xi) $x^4 - (5x + 6)^2 = 0$.
(xii) $x^4 + x^3 - x - 1 = 0$.
(xiii) $(x^2 - 9)^2 - 4x^2 + 36 = 0$.
(xiv) $(x^2 - 3x - 8)^2 + 2(x^2 - 3x - 8) \quad 8 = 0$.
(xv) $2(x^2 - x - 2)^2 + 5(x^2 - x - 2)(x + 1) - 3(x + 1)^2 = 0$.

8. One root of each of the following equations is given. Find the others :—

(i) $x^3 - 8x^2 + 11x + 20 =$; given root, $+ 4$.
(ii) $x^3 + 7x^2 - 4x - 28 = 0$; given root, $+ 2$.
(iii) $4x^3 - 31x + 15 = 0$; given root, $+ \frac{5}{2}$.
(iv) $9x^3 - 73x + 24 = 0$; given root, $+ \frac{8}{3}$.

9. The roots of a cubic equation are (i) $+ 8, - 5, - 3$; (ii) $- 6, + 2, + 4$; (iii) $0, - 3, + 3$. Determine by inspection the values of the coefficient of x^2 and the constant term in each case.

10. A cubic equation has the form $x^3 - px + q = 0$. Show that $p = \alpha^2 + \alpha\beta + \beta^2$ and $q = \alpha\beta(\alpha + \beta)$ where α and β are any two of its three roots. Test the conclusion by applying it to either No. 9 (i) or No. 9 (ii) and to either No. 8 (iii) or No. 8 (iv).

11. The graph of the function

$$y = x^3 + 3x^2 - 4x - 12$$

is moved one unit to the right. Show that the formula corresponding to the new graph contains no term involving x^2.

12. State the relation between the roots of the equations

$$x^3 + 3x^2 - 4x - 12 = 0$$

and $$x^3 - 7x - 6 = 0.$$

13. Draw on the same sheet of squared paper the graphs of $y = x^3$ and of $y = 7x + 6$. Deduce the values of the roots of the equation

$$x^3 + 3x^2 - 4x - 12 = 0.$$

14. Find how the graph of the function

$$y = x^3 - 1{\cdot}5x^2 - 2{\cdot}5x + 3$$

must be moved in order that it may correspond to a formula of the type $$y = x^3 - px + q.$$
Use your result to find the roots of the cubic equation.

$$x^3 - 1{\cdot}5x^2 - 2{\cdot}5x + 3 = 0$$

by the graphic method of No. 33. [Use the same graph of $y = x^3$.]

15. Find by means of the graph of $y = x^3$ the roots of the following cubic equations:—

(i) $x^3 + 3{\cdot}5x^2 - 7x - 20 = 0.$

(ii) $x^3 - x^2 - 0{\cdot}72x + 0{\cdot}576 = 0.$

EXERCISE XLV.

INVERSE PARABOLIC FUNCTIONS (I).

A.

1. A marble is rolled several times up a sloping board 2 metres long. The slope of the board is altered for each experiment, and in the several experiments the marble starts with a different velocity and from a different point of the board. From the following data obtain formulæ for the distance (S) of the marble from the upper end of the board t seconds after projection. Determine in each case the highest point reached by the marble and the moment when it reaches that point. The velocity is measured in centimetres per second.

(i) Starting-point, 40 cms. from lower end; $v = +20 - 2t$.
(ii) Starting-point, the middle; $v = 21 - 3t$.
(iii) Starting-point, 8 cms. from lower end; $v = +48 - 6t$.
(iv) Starting-point, 50 cms. from lower end; $v = 40 - 5t$.

What is the interpretation of the last result?

How could the moments of turning be foretold from the formulæ for the velocity?

2. Change the subject to t in each of the formulæ for S in No. 1. Use the formulæ to determine in each case the moments (i) when the marble crosses the middle line of the board; (ii) when the marble falls over the lower or upper end of the board.

How can the highest points reached by the marble be deduced from these formulæ?

3. The length of a cricket pitch is 22 yards. The path followed by the centre of a ball from a certain bowler's hand is described by the formula:—

$$h = 71{\cdot}68 + 0{\cdot}16d - 0{\cdot}02d^2$$

h being the height of the ball above the ground in inches,

d the horizontal distance in feet from the "bowling crease" (the line in which the wickets at the bowler's end are inserted). The vertical plane which contains the path of the ball also passes through the middle wicket at each end. Calculate the greatest height of the centre of the ball, its height when it leaves the bowler's hand immediately above the bowling crease, and its height when it reaches the plane of the opposite wickets. Explain the meaning of the first and third results. Sketch the path of the ball.

4. Obtain a formula for the distance of the ball from the bowling crease when its centre is at a given height. Use it to find where the ball (which has a diameter of 2·9 inches) would hit the ground if allowed to do so. The batsman hits it when its centre is 1 foot from the ground; how far is it then from his wicket?

5. When struck by a batsman a cricket ball follows the path indicated by the formula :—

$$h = 0{\cdot}597 + 0{\cdot}594d - 0{\cdot}003d^2$$

h and d being both measured in feet. Find the greatest height reached by the ball.

6. Change the subject of the last formula to d. The batsman is caught out by a fielder who catches the ball when it is 6 feet from the ground. Where may he have been? What would have been the horizontal range of the ball if it had not been caught?

7. The first term of an A.P. is + 29 and the common difference − 2. Write a formula for S, the sum of n terms. For what value of n is the sum highest? Why does it decrease when more terms are added?

8. Write a formula for the number of terms of the foregoing series required to yield a given sum. For what numbers of terms is the sum (i) 200, (ii) − 64? Explain the double result in each case.

9. Write a formula for the number of terms of the series

$$+ 16 + 14 + 12 + \ldots$$

required to yield a given sum. Apply it to determine n when S = + 72. Explain the result.

10. Show by arguments based upon the formulæ both for S and for n that the sum of the series in No. 9 cannot be higher than $72\frac{1}{4}$. What is actually its highest value? Why is it not actually $72\frac{1}{4}$?

11. Find in two ways a number above which the sum of

the following series cannot rise. What is actually the highest value of the sum and what is the number of terms which gives it? Explain the result.

$$+\,19 + 16 + 13 + \ldots$$

12. Find how many terms of the foregoing series will yield (i) $+\,30$, (ii) $-\,22$. Are all your results valid? If not, which must be rejected and for what reasons?

13. Find in the easiest way you know the number of terms for which the sum of the series

$$-\,81 - 77 - 73 - \ldots$$

reaches its lowest value. What is that value?

14. Write a formula for the number of terms of the preceding series which must be taken in order to yield a given sum. Find whether the following numbers are possible values of the sum, and, where they are possible, find the corresponding number of terms. Explain why some of the results are impossible.

(i) $S = -231$, (ii) $S = -690$, (iii) $S = +42$, (iv) $S = +174$.

15. Change the subject of the formula

$$S = \tfrac{1}{2}n\,\{2a + (n-1)d\} \text{ to } n.$$

Show that $S > -(2a - d)^2/8d$.

B.

16. Write down relations expressing functions of x which are respectively inverse to the functions given in the following relations:—

(i) $y = 3x - 2$.	(ii) $y = 2{\cdot}8 - 0{\cdot}7x$.
(iii) $7x - 4y + 8 = 0$.	(iv) $3x + 12y - 5 = 0$.
(v) $y = \dfrac{3}{x-2}$.	(vi) $y = \dfrac{4}{2x-3} + 7$.
(vii) $(2x - 5)(3y + 4) = 1$.	(viii) $8xy + 2x - 20y - 5 = 0$.
(ix) $y = 3x^2 + 7x - 1$.	(x) $y = -2x^2 + x \quad 8$.

17. Find the inverse of the linear function $ax + b$. Show that the graphs of a linear function and its inverse will always intersect on the line through the origin which bisects the angle between the axes. Verify by drawing on one sheet the graphs of No. 16 (i), (ii), and (iii) and their inverse functions.

18. Show that the property described in No. 17 holds good also between any hyperbolic function $a/(x + b) + c$ and its inverse. Verify by drawing on the same sheet the graphs of No. 16, (v) and (vi) and their inverse functions.

19. What (if any) are the limits to the possible values of (*a*) the variable, (*b*) the function in No. 16 (ix) and (x)? Answer the same questions with reference to the corresponding inverse functions.

20. Find the inverse of each of the following functions:—

(i) $2x^2 - 3x + 4$.
(ii) $7 + 3x - 5x^2$.
(iii) $ax^2 + bx + c$.
(iv) $2x/(3x - 4)$.
(v) $ax/(bx + c)$.
(vi) $3x^2/(2x - 1)$.
(vii) $(2x^2 - 3x + 5)^{-1}$.
(viii) $x(x - 1)^{-2}$.
(ix) $\sqrt{(2x + 3)}$.
(x) $\sqrt{x} + \sqrt{(x - 2)}$.

21. Show that the product $3x(2x - 5)$ is positive if x is above $+ 2\cdot5$ or below zero, but negative if $+ 2\cdot5 > x > 0$. Show also that the product $(3x + 2)(2x - 5)$ is positive only if x is above $+ 2\cdot5$ or below $- \frac{2}{3}$.

22. What is the range of possible values of the variable in each of the following functions? Are there any limits to the value of the function?

(i) $\sqrt{x^2 - 5x}$.
(ii) $\sqrt{(3x^2 + 2x)}$.
(iii) $\sqrt{(2x - 1)(x + 3)}$.
(iv) $\sqrt{(2x^2 + 11x - 21)}$.
(v) $\sqrt{\{(x + 7)/(x - 3)\}}$.
(vi) $\sqrt{\{(x^2 - 4)/(8 - 3x)\}}$.

23. Show that the function of No. 20 (vi) has no values between 0 and + 3. Show that the function assumes these values when $x = 0$ and $x = + 1$ respectively.

24. Determine the sign of the function $3x^2/(2x - 1)$ when x is a very little below and again when it is a very little above zero. What peculiarity of the value when $x = 0$ is implied by your results? Examine in a similar way the sign of the function for values of x a little below and above + 1. What do you conclude about the value when $x = + 1$?

25. Sketch the graph of the function $3x^2/(2x - 1)$ from $x = - 1$ to $x = + 3$. (Pay careful attention to the values of the function a little below and above $+ 0\cdot5$.)

26. Show that the function in No. 20 (viii) has a turning value of $- 0\cdot25$. Is this a higher or a lower turning value? What is the corresponding value of x? What statements, corresponding to these, may be made about the inverse function? Sketch the graph of the function from $x = - 1$ to $x = + 2$ and the corresponding part of the inverse function.

27. What are the turning values of the functions inverse to No. 22 (iii) and (iv)? Sketch the two functions and the corresponding inverse functions.

28. Find the function inverse to No. 22 (v). Sketch the graph of the direct and inverse functions. Has either of them turning values?

29. Find the inverse of the function in No. 22 (vi). Sketch the graph of the direct and inverse functions.

30. Find the turning value of the function in No. 20 (vii). Sketch the graph of the function and of its inverse.

EXERCISE XLVI.

INVERSE PARABOLIC FUNCTIONS (II).

A.

1. Show algebraically that the function $x^2 - 2cx \cos \alpha + c^2$ can be expressed in the form $(x - c \cos \alpha)^2 + c^2 \sin^2 \alpha$. Hence show (i) that it is always positive and (ii) has a lower turning value. (It is to be assumed that α is constant as well as c.)

2. Illustrate the results of No. 1 from the geometry of the triangle, distinguishing between cases in which α is (i) $< 90°$, (ii) $> 90°$.

3. Use the identity of No. 1 to change the subject of the formula $a^2 = b^2 + c^2 - 2bc \cos \alpha$ to b. Is there any limit to the values of a which will yield values of b? How many values of b correspond to a given value of a? In what case is there only one value of b (or two equal values)?

4. Two straight roads diverge at an angle of 42°. A treasure is hidden in the hedge of one road and is known to be exactly 100 yards from a tree which grows in the hedge of the other. The tree is 120 yards from the corner where the two hedges meet. Calculate the distances from this corner at which the treasure may be found and show the positions in a diagram.

5. One dark night a ship foundered in the English Channel. A coastguard on a cliff, noting the interval between seeing the flash of a gun fired on board and hearing the report, estimated that it was 1 mile from shore. At the coastguard station $1\frac{1}{2}$ miles east of this cliff the flash of the gun was observed to bear 53° W. of S. but the report was inaudible. Draw a diagram showing the two places in which the ship may have sunk and calculate their distances from the coastguard station.

Note.—Nos. 1-5 show that if two sides of a triangle are

given and the angle *opposite* to one of them, the length of the third side cannot always be calculated with certainty. The case is **ambiguous.**

6. Show from a consideration of the result of No. 3 that if A is obtuse b has only one possible value.

7. Given $a = 92$, $c = 120$, $A = 36°$ calculate the possible values of b and of the angles B and C.

B.

8. In the function $\sqrt{(b^2 + c^2 - 2bc \cos \alpha)}$, b and c are constant and non-directed but α varies from 0° to 360°. Show that the value of the function ranges between $b - c$ and $b + c$. Illustrate by the following figure: Draw b (assumed to be greater than c) from O along the x-axis. With O as centre draw a circle of radius c. Let α be the angle between b and c in its various positions. What lines give the various values of the function?

9. Draw a circle of radius r with centre C. Take any point O outside the circle and let OC $= d$. Draw from O a straight line OP cutting the circle at P and making an angle α with OC. Show that l, the length of OP, is given by the quadratic equation

$$l^2 - 2ld \cos \alpha + d^2 - r^2 = 0.$$

What is the product of the two values of OP which correspond to a given value of α? How does the equation show that their product is equal (for all possible values of α) to the square of the tangent to the circle from O? How does the equation show that $\sin \alpha$ cannot be greater than r/d?

10. Take the point O within the circle. Show that the product of the two values of OP (corresponding to a single value of α) is now negative. What is the meaning of this result? What geometrical truth does the equation demonstrate?

11. In a straight line of length a a point is taken x from one end. What is the product of the two segments of the line? Show that it has its greatest value when the line is bisected.

12. Find the sum of the squares of the two segments of the line in No. 11. Show that it can be expressed in the form

$$\frac{a^2}{4} + 2\left(x - \frac{a}{2}\right)^2$$

Hence show that the sum of the squares is least when the line is bisected.

13. The sine of an angle is $\frac{5}{13}$. Show algebraically that two cosines and two tangents correspond to this value. Exhibit in a figure the possible values of the angle.

14. The tangent of an angle is 16/63. Find the corresponding values of the sine and cosine and exhibit in a figure the possible values of the angle.

15. The numbers -12, $+16$, and -15 are said to be proportional respectively to the sine, cosine, and tangent of a certain angle. Is this the case, and, if so, what is the value of the angle?

EXERCISE XLVII.

AREA FUNCTIONS.

A.

1. Write down the area functions which correspond to the following ordinate functions:—

(i) $y = 2x^2$. (ii) $y = 1{\cdot}8x^2$.
(iii) $y = -0{\cdot}9x^2$. (iv) $y = 6\sqrt{x}$.
(v) $y = 1{\cdot}5\sqrt{x}$. (vi) $y = 5{\cdot}1\sqrt{(-x)}$.

2. Calculate the area between the curve, the x-axis and the ordinate specified in the following cases taken from No. 1:—

The ordinate $x = +6$ in (i); $x = +10$ in (iii); $x = +25$ in (iv); $x = -100$ in (vi); $x = -8$ in (ii).

3. In No. 1 (i) calculate the area between the curve, the x-axis, and the ordinates $x = +2$ and $x = +9$.

4. In No. 1 (ii) calculate the area between the curve, the x-axis, and the ordinates $x = -6$ and $x = +10$.

5. In No. 1 (v) calculate the area between the curve, the x-axis, and the ordinates $x = +4$ and $x = +49$.

6. Fig. 43, A and C (p. 233), are parabolic. Calculate the area above the x-axis in each figure, taking a small square of the chequered background as the unit.

7. In fig. 43, C, calculate the area of the band between two horizontal straight lines respectively 9 and 16 scale-units below the vertex.

8. In fig. 43, B, calculate the area included between two horizontal straight lines drawn respectively 5 and 10 scale-units above the x-axis.

9. In fig. 44 measure a number of vertical lengths, such as pq, intercepted between the curve AB and the line AC. Show that they follow the law $pq = kx^2$ (where $x = \text{A}m$) and find the value of k. What is the area of the space included between AB, AC, and BC? (Compare Ex. XXX, Nos. 31, 32.)

10. In fig. 44 what is the area of ABD and of ABE?

11. Give an expression for the total height of an ordinate such as pm at distance x from A. Write down the corresponding area function, and show that the answer to No. 10 can be obtained from it.

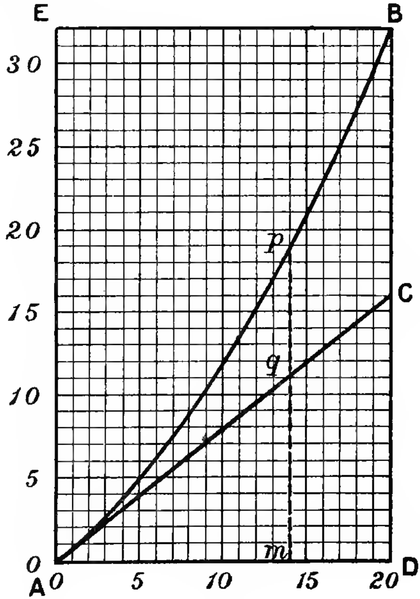


FIG. 44.

12. Each of the following is the ordinate function of a curve. Give the corresponding area functions :—

(i) $y = 3x + 2{\cdot}1x^2$.
(ii) $y = 1{\cdot}6x - 0{\cdot}9x^2$.
(iii) $y = 12 + 4x + x^2$
(iv) $y = 10{\cdot}8 - 2{\cdot}4x + 1{\cdot}5x^2$.
(v) $y = 2x^2 - 3x + 7$.
(vi) $y = \frac{1}{2}x^2 + 5x - 8$.
(vii) $y = 1 + 6\sqrt{x}$.
(viii) $y = 5{\cdot}2x - 4{\cdot}5\sqrt{x}$.
(ix) $y = 1{\cdot}5x^2 - 1{\cdot}2\sqrt{x}$.
(x) $y = 0{\cdot}6 - 1{\cdot}2\sqrt{x} + 2{\cdot}4x - 4{\cdot}8x^2$.

13. Calculate the area under the curve in No. 12 (i) from $x = 0$ to $x = +10$; in No. 12 (iii) from $x = -5$ to $x = +5$; in No. 12 (viii) from $x = +4$ to $x = +36$; in No. 12 (ix) from $x = +9$ to $x = +16$.

Draw graphs to illustrate the second and the last of these problems.

14. Draw the smooth curve passing through the points whose co-ordinates are given in the following table. Find by the method of differences the formula of the curve. Calculate the area under it from $x = 0$ to $x = +20$:—

x : 0	4	8	12	16	20
y : 20	32	36	32	20	0

B.

15. A solid figure is made by taking n circular discs 1 inch thick and, respectively, 1, 2, 3 . . . n inches in radius. The largest disc is placed on the table, the next largest on top of it, and so on in order. A pin is driven through the centres of the discs to keep them together, and projects 1 inch above the highest. Another figure is made by piling $n + 1$ discs, each n inches in radius and 1 inch thick, so that they form a cylinder $n + 1$ inches high. What fraction is the volume of the former solid of the volume of the cylinder? Calculate the actual volume when $n = 10$.

16. What formula in mensuration is obtained by supposing the discs in No. 15 to be increased in number without limit?

17. The angle between the hypothenuse and the perpendicular of a right-angled triangle is 35°. The triangle is made to generate a cone by revolving about the perpendicular. What is the area of the circular section situated at distance h from the vertex? What is the volume of the cone above this section?

18. I have a wooden model in the form of a truncated pyramid upon a square base. It is 10 inches high, the length of the side of the base is 23 inches, the length of the side of the top 3 inches. Write formulæ (i) for the length of the side of a square section at a given distance from the top; (ii) for the area of that section. From the latter formula deduce a formula for the volume of the solid above a given section. By means of this formula calculate the volume of the model.

19. The base of a village cross consists of $m + 1$ square slabs of stone each 1 foot thick. The side of the top slab measures a feet, and the side of each of the others is b feet longer than that of the one above it. Show that the total volume is given by the formula :—

$$V = (m + 1)\{a^2 + mab + \tfrac{1}{6}m(2m + 1)b^2\}.$$

20. Draw the positive branch[1] of the curve

$$y = 12 - 3\sqrt{x}$$

from where it crosses the y-axis to where it crosses the x-axis. Imagine it to generate a solid by revolution about the x-axis. Write formulæ (i) for the area of the section of the solid at the distance x from the origin; (ii) for the volume of the solid from the base up to this section. Deduce the total volume of the solid.

C.

21. A number m of rectangles of equal breadth are set side by side. Their areas are $1^3, 2^3, 3^3, \ldots m^3$. Each lies upon a rectangle which is of the same height and width as the one whose area is m^3. An additional rectangle of this size is placed next before the rectangle under the first of the increasing series. Thus the $(m + 1)$ underlying rectangles form together a rectangle of area $(m + 1)m^3$ Find, by co-operation with the rest of the class, what fraction of this last rectangle is covered by the increasing series when m has the values $1, 2, 3, \ldots 10$. What laws are followed by the numerators and denominators of these fractions?

22. The increasing rectangles are made so thin and at the same time so numerous that they become indistinguishable from the area under the curve $y = kx^3$. Show that the formula for this area, from the origin up to the ordinate x from the origin is $A = \frac{1}{4}kx^4$. What assumption have you made in this proof?

23. OP is a portion of the curve $y = kx^3$ in the first quadrant, O being the origin. From P perpendiculars are drawn, PM to the x-axis, PA to the y-axis. Show that the area AOP is three-fourths of the rectangle AM.

[1] That is, the curve obtained by taking only the positive square roots of the successive values of x.

24. From the result of No. 23 deduce the area function of the curve whose ordinate function is $y = k\sqrt[3]{x}$.

25. How can the result of No. 24 be brought under Wallis's Law?

26. Write down the area functions corresponding to the following ordinate functions :—

(i) $y = 2{\cdot}4x^3$.
(ii) $y = -3{\cdot}2x^3$.
(iii) $y = 1{\cdot}2\sqrt{x}$.
(iv) $y = -8\sqrt[3]{x}$.
(v) $y = 1 - 2x^3$.
(vi) $y = 1 + 2x + 3x^2 + 4x^3$.
(vii) $y = 3{\cdot}2 - 1{\cdot}4x + 2{\cdot}7x^2 - 3{\cdot}2x^3$.
(viii) $y = 4 - 12\sqrt[3]{x}$.
(ix) $y = (1 + x^3)/(1 + x)$.
(x) $y = (1 - x^4)/(1 - x)$.
(xi) $y = (1 - x^4)/(1 + x)$.
(xii) $y = (16 - 81x^4)/(2 - 3x)$.

27. Calculate the areas under the following curves in No. 26 :—

(i) from $x = 0$ to $x = +6$.
(ii) from $x = 0$ to $x = -10$.
(iv) from $x = -10$ to $x = +10$.
(vi) from $x = -4$ to $x = +4$.
(x) from $x = -5$ to $x = +5$.
(xi) from $x = -5$ to $x = +5$.

Draw on the same sheet the curves of No. 26 (vi), (x), and (xi) from $x = -5$ to $x = +5$.

28. Write out the proof by recurrence that the fraction

$$\frac{0^2 + 1^2 + 2^2 + \ldots + m^2}{(m+1)m^2} = \frac{1}{3} + \frac{1}{6m}$$

for all values of m.

Note $\Sigma_o^m m^2 \equiv 0^2 + 1^2 + 2^2 + 3^2 + \ldots + m^2$.

29. Find by the method of differences the formula for $\Sigma_o^m m^2$.

30. Prove by the recurrence method that

$$\Sigma_o^m m^3/(m+1)m^3 = \frac{1}{4} + \frac{1}{4m}$$

for all values of m. (See Ex. XXXVIII, No. 10.)

EXERCISE XLVIII.

DIFFERENTIAL FORMULÆ.

A.

1. Find in each of the following cases the first difference of y corresponding to any increment h in the value of x. Deduce from your results the corresponding differential formulæ of the first order:—

(i) $y = 3x - 11$.	(ii) $y = (a - b)x + (a + b)$.
(iii) $y = 4{\cdot}1x^2$.	(iv) $y = 4{\cdot}1x^2 - 3{\cdot}7x + 1{\cdot}8$.
(v) $y = 7x^3$.	(vi) $y = 1/2x$.

2. Use Wallis's Law to derive differential formulæ of the first order from the following primitives:—

(i) $y = \frac{1}{4}x - \frac{5}{3}$.	(ii) $y = 4{\cdot}8 - 3{\cdot}5x$.
(iii) $y = 2{\cdot}7x^2$.	(iv) $y = 4{\cdot}3x^2 - 1{\cdot}2x + 5{\cdot}8$.
(v) $y = 8 + 12x - 5x^2$.	(vi) $y = 2x^3$.
(vii) $y = x^3 - 4x + 3$.	(viii) $y = 4x^3 + x^2 - 7x + 1$.
(ix) $y = 3x + 1 + 1/3x$.	(x) $y = (5x^2 - 7)/x$.

3. Use Wallis's Law to obtain the primitive of each of the following differential formulæ:—

(i) $\delta y/\delta x = 2$.	(ii) $\delta y/\delta x = -\frac{1}{3}$.
(iii) $\delta y/\delta x = 8x$.	(iv) $\delta y/\delta x = 2x - 3$.
(v) $\delta y/\delta x = 5 - 3x$.	(vi) $\delta y/\delta x = 6x^2 - 4x + 1$.
(vii) $\delta y/\delta x = 3/x^2$.	(viii) $\delta y/\delta x = 1 - 1/x^2$.

4. The primitives of No. 3 are satisfied respectively by the following pairs of values of x and y. Use this information to determine the values of the unknown constants in them:—

(i) $(-7, +5)$.	(ii) $(+4, 0)$.
(iii) $(0, 0)$.	(iv) $(-2, +5)$.
(v) $(+4, -3)$.	(vi) $(+1, +1)$.
(vii) $(-3, +3)$.	(viii) $(+2, +2{\cdot}5)$.

5. Derive a differential formula of the second order from each of the following:—

(i) $y = \frac{1}{8}x^2 - 2x + 3$. (ii) $y = 5x^3 - 3x^2 + 14x - 11$.
(iii) $y = 2(x - 1)^3$.

6. Find the primitive of each of the following differential formulæ :—

(i) $\delta^2 y/\delta x^2 = -3$, given that it is satisfied by $(-2, +5)$ and $(0, +1)$.
(ii) $\delta^2 y/\delta x^2 = 6x - 2$, given that it describes a curve which passes through the points $(+1, 0)$ and $(-1, -4)$.
(iii) $\delta^3 y/\delta x^3 = 12$, given that its graph passes through the points $(0, 0)$, $(+1, -9)$, $(-1, -13)$.

7. Prove that, when $y = kx^4$, $\delta y/\delta x$ can be calculated in accordance with Wallis's Law.

8. Write down the primitive formulæ which correspond to :—

(i) $\delta y/\delta x = ax^3$. (ii) $\delta^2 y/\delta x^2 = 2x^2 - 3x + 1$.
(iii) $\delta^3 y/\delta x^3 = 5x$. (iv) $\delta^4 y/\delta x^4 = -1$.

B.

9. Given that y is a function of x of the nth degree (where n may be either 1, 2, 3, or 4), show that the differential formula of the nth order expresses an exact equality for all values of δx.

10. A variable y is connected with x by the relation $y = 5x^2$. Show that if the true value of $\delta y/\delta x$ is to differ by less than 0·001 per cent from the value expressed by the differential formula δx must represent a number less than $|2x| \times 10^{-5}$.

11. The range of values over which the relation of No. 10 holds good is (i) from $x = +1$ upwards; (ii) from $x = +10{,}000$ upwards; (iii) from $x = +0{\cdot}01$ upwards and from $x = -0{\cdot}01$ downwards; (iv) from $x = +10^{-5}$ upwards and $x = -10^{-5}$ downwards. Find in each of these cases the largest numerical value of the increment of x which can be called "small".

12. "The differential formula $\delta y/\delta x = 10x$, derived from $y = 5x^2$, can (by taking $|\delta x|$ small enough) be regarded as true to any required degree of accuracy for all values of x except $x = 0$." Justify this statement.

Note.—As x approaches zero (either from the positive or the negative side) $\delta y/\delta x = 10x$ also approaches zero. By taking x near enough to zero, $\delta y/\delta x$ may thus be made numeri-

cally smaller than any number which can be named. For this reason, although (as was shown in No. 12) the differential formula cannot really be applied when $x = 0$, it is usual to say that " $\delta y/\delta x = 0$ when $x = 0$ ".

13. A variable y is connected with x by the relation $y = 5x^2 + 3x - 7$ from $x = 0$ upwards. In a certain calculation it is necessary that the true value of $\delta y/\delta x$ should not differ by more than one-thousandth part from the value given by the differential formula. What is the greatest value of the increment of x which can be symbolized by δx?

14. Show that when $y = 5x^2 + 3x - 7$ the differential formula $\delta y/\delta x = 10x + 3$ may be made to hold good to any required degree of accuracy for all values of x except $x = -0{\cdot}3$. Explain carefully what is meant when it is said that $\delta y/\delta x = 0$ for this value of x.

Note.—The investigation covered by Nos. 11-14 was necessary to make clear what is meant by the statement that $\delta y/\delta x = 0$ for a certain value of x. When this statement is understood the value of x in question is most easily determined by substituting zero for $\delta y/\delta x$ in the differential formula. Thus, if $y = px^2 + qx + r$, $\delta y/\delta x = 2px + q$. Hence $\delta y/\delta x = 0$ when $2px + q = 0$, that is, when $x = -q/p$. That is to say, $\delta y/\delta x$ may be brought as near as we please to zero by bringing x sufficiently near to $-q/p$, although the differential formula, strictly speaking, fails when $x = -q/p$ exactly. Nos. 12 and 14 show that it cannot be applied in this case because the condition that the formula shall be even approximately true is that the increment of x shall be zero—an obvious contradiction of the meaning of the word "increment".

15. Find the values of x (if there are any) for which $\delta y/\delta x = 0$ in the following cases:—

(i) $y = 3x^2 - 4x - 1$. (ii) $y = 6 - 2{\cdot}3x$.
(iii) $y = 12{\cdot}3 + 4{\cdot}5x - 1{\cdot}5x^2$. (iv) $y = 2x^3 + 3x^2 - 36x + 7$.
(v) $y = \frac{2}{3}x^3 - \frac{11}{2}x^2 + 15x - 2$.

16. Find the values of x (where there are any) for which $\delta^2 y/\delta x^2 = 0$ in the case of the relations specified in No. 15.

EXERCISE XLIX.

GRADIENTS.

A.

Note.—On the curve PT (fig. 45) take any two points PP′. Through PP′ draw the secant PS. It is evident that P′ can always be taken so near to P that the curve between the two points nowhere departs from the line PS by more than a certain distance (say d) which may be chosen as small

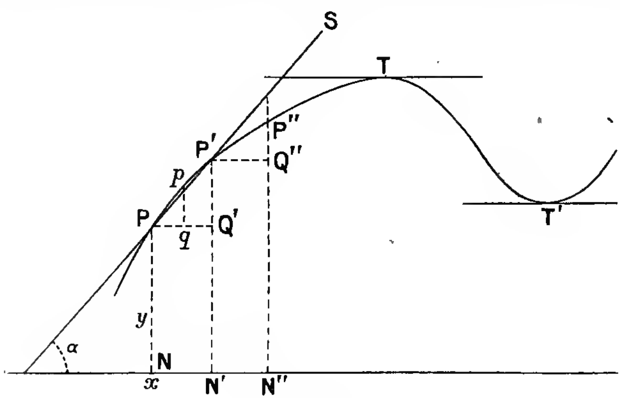


Fig. 45.

as we please. Another way of stating this same fact is to say that if p is any point on the curve between P and P′ the ratio $pq/\mathrm{P}q$ can be made to differ from the ratio P′Q′/PQ by less than a number (say ϵ) which may be taken as small as we please. This being the case, PQ′ or any part of it, such as Pq, may be called δx and the corresponding first difference of the ordinates, i.e. P′Q′ or pq, may be called δy.

Thus the ratio $P'Q'/PQ$ (or any of the ratios pq/Pq) is the value of $\delta y/\delta x$ at the point P. But $P'Q'/PQ = \tan \alpha$, α being the inclination of PS to the x-axis. Now if PS were the tangent to the curve at P, $\tan \alpha$ would be called the **gradient** of the curve at the point P. It follows that, in the circumstances described, the value of $\delta y/\delta x$ for the point P will give, to as close an approximation as anyone chooses to name, the gradient at that point.

If P were taken close to and on the left of the upper turning point T, $\tan \alpha$, and therefore $\delta y/\delta x$, would be positive but would approach zero as P approached T. Again, if P were close to T on the right, $\tan \alpha$, and therefore $\delta y/\delta x$, would be negative and would approach zero as P moved towards T. At the point T itself $\delta y/\delta x$ may be said, then, to be zero according to the convention explained in the notes to Nos. 13 and 15. The value of $\delta y/\delta x$ would obviously exhibit similar changes if P moved from left to right through the lower turning-point, T′, except that the change of sign would be from *minus* to *plus* instead of from *plus* to *minus*.

An upper turning-point in a curve is usually called a **maximum** and a lower turning-point a **minimum**. These names are also given to the turning values of functions. They will sometimes be used in future examples.

1. Calculate the gradients of the curve $y = x^3 - 3x + 2$ at the points where x has the values (i) -2; (ii) -1; (iii) 0; (iv) $+1$; (v) $+2$. Determine whether the turning-points are maxima or minima by considering the signs of the gradients on each side of them.

2. Verify the results of No. 11 by drawing the graph of the curve from $x = -2{\cdot}2$ to $x = +2{\cdot}2$. The part near the y-axis must be drawn with special care. At each of the points specified in No. 11 draw a straight line having the calculated gradient. Determine whether these lines appear to be tangents.

3. Calculate the gradients of the curve

$$y = 2x^3 + 3x^2 - 36x + 7$$

at the points where x has the values (1) $-3{\cdot}5$; (ii) -3; (iii) -1; (iv) 0; (v) $+1$; (vi) $+2$; (vii) $+2{\cdot}5$. Determine whether the turning-points are maxima or minima.

4. Verify the results of No. 2 by drawing the graph of the curve from $x = -4$ to $x = +3$. It will be convenient to make the vertical ten times smaller than the horizontal

scale. The portion between $x = -1$ and $x = 0$ must be drawn with special care. At each of the points specified in No. 2 draw a straight line having the calculated gradient, making allowance for the fact that the horizontal and vertical scales are different. Determine whether the lines so drawn appear to be tangents to the curve.

5. Show that a function of the form $px^3 + qx + r$ cannot have turning values unless p and q are of opposite signs.

6. Find the condition that a function of the form

$$px^3 + qx^2 + rx + s$$

may have turning values.

7. Which of the following curves have turning-points? Where they exist find their co-ordinates :—

(i) $y = 3x^2 - 4x + 8$; (ii) $y = 7 \cdot 1 - 2 \cdot 3x - 4 \cdot 6x^2$;
(iii) $y = x^3 + 3x^2 + 9x + 5$; (iv) $y = 4x^3 - 15x^2 - 18x + 11$;
(v) $y = x^4 - x^3 + x^2 - 1$.

B.

Note.—In fig. 45 the curve in the neighbourhood of P is below the tangent or concave towards the x-axis. In this case it is evident that the differences of y, P′Q′, P″Q″, etc., must decrease towards the right of P and that the corresponding differences towards the left of P must also decrease. In other words, $\delta^2 y$ must be negative for the value of x at P whether the small differences in x are measured to the right or left. Now suppose that $\delta^2 y/\delta x^2 = R$, where R may be either a constant number or an expression (such as $5x - 2$) whose value depends upon x. Since it can be written

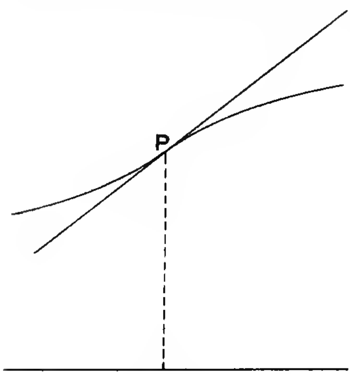


FIG. 46.

$\delta^2 y = R(\delta x)^2$ the sign of $\delta^2 y$ must be the same as the sign of R whether δx is positive or negative—i.e. whether the equidistant ordinates are drawn to the right of P or to the left. If, then, the curve is below the tangent, R, and therefore

$\delta^2y/\delta x^2$, must be negative. Similarly, if the curve is above the tangent, as at T′, $\delta^2y/\delta x^2$ will be positive.

Sometimes it will happen that the curve lies above the tangent on one side of P and below it on the other—as in fig. 46. In this case $\delta^2y/\delta x^2$ will be positive at any point close enough to P on the one side and negative at any point close enough to P on the other side. At P itself, therefore, $\delta^2y/\delta x^2$ may be considered as being zero in the sense explained in the note on p. 257. Such a point as P in fig. 46 is called a **point of inflexion**.

Thus to find the situation of the curve with respect to the tangent at any point it is best to determine the value of $\delta^2y/\delta x^2$. If this is positive the curve is above the tangent, if negative, below. If, as given by the formula, $\delta^2y/\delta x^2 = 0$ we must examine its value for points immediately to the right and left of the given point. If the signs of these values are different the point is a point of inflexion.

8. Examine the value of $\delta^2y/\delta x^2$ at each of the points mentioned in No. 1, and so determine which are turning-points or points of inflexion. Compare the results of your calculation with the graph.

9. Repeat this investigation upon the data of No. 3.

10. Show that the curve $y = 3x^4 - 20x^3 + 48x^2 - 36x + 11$ has two points of inflexion.

11. Examine the curves of No. 7 for points of inflexion.

12. Show that no curve of the type $y = px^2 + qx + r$ can have a point of inflexion.

13. Show that all cubic curves of the type $y = px^3 + qx^2 + rx + s$ have one point of inflexion. Show also that in those of the type $y = px^3 + qx + r$ the point of inflexion always lies on the y-axis. (See the graph of No. 1.)

14. Show that the curve $y = px^4 + qx^3 + rx^2 + sx + t$ has either two points of inflexion or none. Show that if q is zero the points of inflexion (if they exist) are equidistant from the y-axis.

C.

15. A rectangular concert hall AC (fig. 9, p. 34) is to be fitted with a platform FH. AD must be 50 feet, but the length AB is not fixed. FG is to be double of EF and EF is to be

one-tenth of the length (l) of the hall. Show that the area not covered by the platform is given by the formula

$$A = 50l - \tfrac{1}{5}l^2.$$

Hence show that this area will be greatest if the hall is 125 feet long. Calculate the area in this case and verify that $l = 120$ and $l = 130$ both give smaller values. [Apply the methods explained in the notes before Nos. 1 and 8 to find the turning value of the function $50l - \frac{1}{5}l^2$ and to show that it is a maximum. The $\delta y/\delta x$ and $\delta^2 y/\delta x^2$ of the notes will be replaced here by $\delta A/\delta l$ and $\delta^2 A/\delta l^2$.]

16. Fig. 17 (p. 35) is the surface of a plate whose thickness is c. Supposing that b is always twice c show that the volume of the plate is given by the formula

$$V = (3a - 4b)c^2.$$

Show that if a is a fixed length, but c may be varied, the volume is greatest when $c = a/2$. Verify (as in No. 15) by means of numbers chosen by yourself.

17. Fig. 18 (p. 35) is the surface of a plate whose thickness is b. Show that the weight of the plate is given by the formula

$$W = (3a - 2b)b^2 w$$

where w is the weight of a unit cube of the material. Show that if the length a is fixed but b may be varied the plate will get constantly lighter as b increases from zero to a.

18. Fig. 28 (p. 94) represents a rectangular plate from which two semi-circular pieces have been removed. Show that if a is fixed but b may be varied, the area is greatest when $b = 3a/2\pi$.

19. Fig. 28 may be taken as a central section of a block of wood out of which two hemispherical pieces have been cut of radius b. The height of the block is $2(b + c)$. Thus two of the sides perpendicular to the plane of the paper are rectangles measuring a by $2(b + c)$ but containing circular holes of radius b. Show that the volume of the block is given by

$$V = 6acb + 6ab^2 - \tfrac{4}{3}\pi b^3.$$

Show that if a and c are fixed but b is variable the maximum and minimum values of the volume correspond to the values of b which satisfy the equation

$$4\pi . b^2 - 12a . b - 6ac = 0.$$

Hence show that there will be neither a maximum nor a minimum value unless

$$c < 3a/2\pi.$$

Calculate the maximum and minimum volumes when $a = 2\pi$ inches and $c = 9$ inches.

20. Fig. 29 (p. 94) represents a star-shaped plate pierced by a circular aperture. Show that if b is fixed and a varies the area of the metal surface constantly increases as a increases and never has a minimum or maximum.

21. A seaside camp offers to receive from a certain school a party of not less than 5 at a cost per head of $(2 + 200/n^2)$ shillings a week, n being the number in the party. Find the number which it would be cheapest for the school to send.

22. The size of parcels which the Post Office will accept for transmission is limited by the regulation that the length and girth of the parcel when added together must not exceed 6 feet. Find the dimensions of the largest box with square ends which it is possible to send through the post.

23. Take a square sheet of paper, PQRS, and draw in the middle of it a square ABCD. Produce the sides of the inner square to meet those of the outer square. Remove, by cutting, the squares AP, BQ, CR, DS. Fold the four wings about the sides of the inner square so as to make a hollow box of which the square is the base. Show that the volume of this box is greatest when AB is two-thirds of PQ.

24. Draw a circle of radius r and inscribe in it a rectangle of length $2a$ and breadth $2b$. Revolve the whole figure about the diameter parallel to the sides of length $2b$ and so generate a sphere with a cylinder inscribed in it. Show that the volume of the cylinder is given by the formula

$$V = 2\pi b(r^2 - b^2).$$

Hence show that the volume of the cylinder is greatest when $b = r/\sqrt{3}$. Calculate the volume in terms of r.

EXERCISE L.

THE CALCULATION OF π AND THE SINE-TABLE.

A.

1. Let $C =$ the length of any chord, AB, of a circle of unit radius (fig. 47) and $C_1 =$ the length of AB′, the chord which bisects the arc AB. Write out a proof that

$$C_1 = \sqrt{\{2 - \sqrt{(4 - C^2)}\}}.$$

2. Take AB as a diameter of the circle and use the formula to calculate the length of side of an inscribed square. Prove independently that your result is correct.

3. Let $E \equiv$ the length of PQ (fig. 48) and $C \equiv$ the length of AB. Write out a proof that

$$E = \frac{2C}{\sqrt{(4 - C^2)}}.$$

4. Use the formula of No. 3 to obtain an expression for the length of side of a square circumscribed about a circle (i) of unit radius, (ii) of radius r.

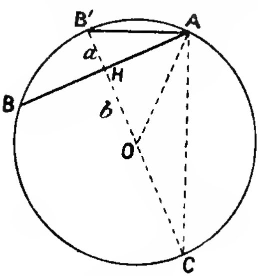


FIG. 47.

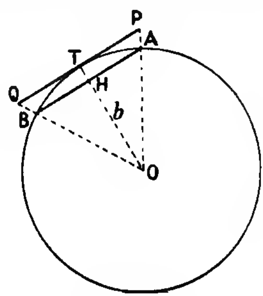


FIG. 48.

5. What is the meaning of the result obtained from the formula of No. 3 by taking AB as the diameter of the circle?

6. Calculate to three places of decimals the perimeter of a regular hexagon circumscribed about a circle of unit radius.

7. Within what limits is the value of π fixed by consideration of the inscribed and circumscribed hexagons?

8. Show that the perimeters of the regular 12-gons which are inscribed within and circumscribed without a circle of unit radius are respectively of length

$$12\sqrt{(2-\sqrt{3})} \text{ and } 24\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}.$$

9. Given that $\sqrt{3} = 1{\cdot}732051$, calculate the limits within which the value of π is fixed by consideration of the inscribed and circumscribed 12-gons.

10. Give expressions for calculating the lengths of the perimeters of the regular octagons inscribed within and circumscribed about a circle of unit radius.

11. Use these expressions to determine major and minor limits for π, given that $\sqrt{2} = 1{\cdot}414213$.

12. Change the subject of the formula of No. 1 to C.

13. Use the new formula to calculate the length of side of an equilateral triangle inscribed in a circle of unit radius.

14. Calculate the length of side of an equilateral triangle circumscribed about a circle of unit radius. What limits are determined for π by this result and that of No. 13?

15. Above each of the points 3, 4, 6, 8, 12 on the x-axis plot a pair of points whose ordinates are respectively the upper and lower limits of the value of π determined by considering the perimeters of regular polygons of 3, 4, 6, 8, 12 sides. Join the two series of points by smooth curves. Determine by extrapolation the probable subsequent course of the curves and the ultimate value of π so far as it can be read from your graph. [Take as wide a vertical scale as possible.]

B.

16. From the results used in No. 15 write down a table giving to three decimal places the values of sin a and tan a when $a = 7\frac{1}{2}°, 22\frac{1}{2}°, 30°, 45°, 60°$. Compare the results with the table on p. 111.

17. Change the formula of No. 1 into a formula for finding $\sin\frac{a}{2}$ where sin a is known.

18. Use this formula to calculate the value of $\sin 3\frac{3}{4}°$. [Take the value of $\sin^2 a$ from No. 9.]

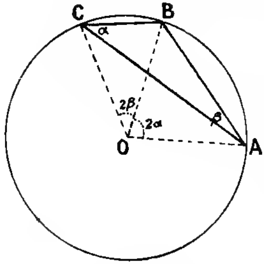


FIG. 49.

19. Show that your value for $\sin 7\frac{1}{2}°$ is approximately twice that obtained from $\sin 3\frac{3}{4}°$. That being the case you may, from the other two sines, calculate sin 5° to three places by proportion. Do so, and deduce from it the value of cos 5° also to three places. Compare the results with the table on p. 111.

20. Use fig. 49 to prove that

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

[This proof can be used only if $\alpha + \beta < 90°$. Why?]

21. Prove by the same method that

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

[In fig. 49 let AOC $= 2\alpha$ and COB $= 2\beta$, then AOB $= 2(\alpha - \beta)$. Produce AB and drop upon it from C a perpendicular CD. Then AB = AD − BD.]

22. From the known values of sin 30°, sin 45°, sin 60°, sin 5° and cos 5°, calculate the values (to three places) of sin 25°, sin 35°, sin 40°, sin 50°, sin 55°, sin 65°. The work is to be divided among the class, each member calculating one or two series only. Collect the results into a table giving the sines of every five degrees from 25° to 65°. Compare this table with the one on p. 111. Of what cosines may this also be considered to be the table?

23. In fig. 47 let $\angle AOB' = 2\alpha$ so that $\angle AOB = 4\alpha$. Prove by the method of No. 20 that

$$\sin 2\alpha = 2 \sin \alpha \cdot \cos \alpha.$$

24. Use the formula to calculate sin 70° and sin 80°, taking your data from the table constructed in No. 22. What cosines have you simultaneously calculated?

25. Give a careful summary of the method by which a complete table of sines and cosines can be calculated for intervals of 5°.

SECTION III.

LOGARITHMS.

EXERCISE LI.

GROWTH FACTORS.

TABLE OF AVERAGE HEIGHTS OF BOYS AND GIRLS.

Age	$5\frac{1}{2}$	$6\frac{1}{2}$	$7\frac{1}{2}$	$8\frac{1}{2}$	$9\frac{1}{2}$	$10\frac{1}{2}$	$11\frac{1}{2}$	years
Boys	41·7	43·9	46·0	48·8	50·0	51·9	53·6	inches
Girls	41·3	43·3	45·7	47·7	49·7	51·7	53·8	,,

Age	$12\frac{1}{2}$	$13\frac{1}{2}$	$14\frac{1}{2}$	$15\frac{1}{2}$	$16\frac{1}{2}$	$17\frac{1}{2}$	$18\frac{1}{2}$	years
Boys	55·4	57·5	60·0	62·9	64·9	66·5	67·4	inches
Girls	56·1	58·5	60·4	61·6	62·2	62·7	—	,,

A.

Note.—Imagine a boy to grow at such a rate that his height at each age is exactly equal to the average height of all boys that age. He may be called, in respect of height, **the average boy.**

1. How much taller does the average boy grow between $6\frac{1}{2}$ and $7\frac{1}{2}$? Between $12\frac{1}{2}$ and $13\frac{1}{2}$? How much does the average girl grow between $10\frac{1}{2}$ and $11\frac{1}{2}$?

Find the ratio of the second of each pair of heights to the first. Which number best measures the growth of the child —the difference of the annual heights or their ratio? Why? Arrange the three annual rates of growth in order of magnitude.

Note.—The ratio of the boy's height on his 11th birthday

to his height on his 10th birthday may be called the **growth-factor of his height at 10 years of age.** How would you find the growth-factor of his height at 12 years of age?

2. Calculate the growth-factor of the average boy's height at $5\frac{1}{2}$. If the growth-factor had the same value at (i) $12\frac{1}{2}$ and at (ii) $14\frac{1}{2}$, how tall would the boy be at $13\frac{1}{2}$ and $15\frac{1}{2}$ respectively? Do these results agree with those given in the table?

3. If the growth-factor of the height of the average girl of $12\frac{1}{2}$ were the same as that of the average boy of the same age, how tall would you expect her to be at $13\frac{1}{2}$?

4. Calculate the growth-factors for the average boy and the average girl at $5\frac{1}{2}$, $9\frac{1}{2}$, $11\frac{1}{2}$, and $14\frac{1}{2}$ years of age. What conclusions do you draw from the results?

5. Suppose the growth-factor of the average girl of $9\frac{1}{2}$ to remain constant for the next few years. Calculate how tall she would be at $10\frac{1}{2}$, $11\frac{1}{2}$, . . . $15\frac{1}{2}$. Compare your results with the table. What conclusion do you draw?

6. Assuming the same growth-factor as in No. 5, calculate what the average height would be at $6\frac{1}{2}$ and at $5\frac{1}{2}$. Compare the results with the numbers given in the table.

7. The population of a town has been increasing for some years at the rate of 20 per thousand per annum. What is the annual growth-factor of the population?

8. The present population of the town of No. 7 is 48,750. What will it be two years hence? What was it two years ago?

9. The population of a country parish is falling off to the extent of 10 per thousand per annum. What is its annual growth-factor?

10. The population of the parish of No. 9 is 4520. What will it be in three years? What was it last year?

11. A quantity is changing in such a way that its magnitudes at equal intervals of time form a geometrical progression with a common ratio r. (In other words, the quantity has for these intervals a constant growth-factor of r.) Its present magnitude is Q_0. Show that its magnitude at the end of any integral number of time-intervals, past or future, is given by the formula

$$Q = Q_0 r^n.$$

12. Write formulæ for the population (P) of the places

mentioned in Nos. 7 and 9 at any exact number of years before or after the present moment.

13. The values of a number, Q, at regular intervals are given by the formula $Q = Q_0 \times (1{\cdot}1)^n$. Draw up a table by which the values could be calculated to within 1 per cent for values of n from -5 to $+5$.

14. Draw up similar tables to deal with the formulæ

$$Q = Q_0 \times (1{\cdot}3)^n \text{ and } Q = Q_0 \times (0{\cdot}8)^n.$$

B.

15. A man's business is increasing so rapidly that his income is doubled regularly in the course of two years. What is the annual growth-factor of his income?

16. Find the annual growth-factor if the income were doubled regularly in the course of four years.

17. What assumption did you make in solving Nos. 15 and 16?

Note.—Nos. 18-20 can be solved by means of the tables of Nos. 13 and 14.

18. Between 1 January, 1906, and 1 January, 1911, the population of a London suburb increased from 5437 to 8758. Assuming the increase to be due to a constant growth-factor find (i) the value of the factor; (ii) the population on 1 January, 1909; (iii) the population on 1 January, 1901.

19. A sapling grew in seven years from a height of 14 inches to a length of 87·8 inches. What constant annual growth-factor would account for this increase?

20. During an epidemic of scarlet fever in London it was noticed that after a certain date the number of cases decreased by the same fraction each week. In the course of eight weeks they fell from 1540 per week to 258 per week. What was the weekly growth-factor? How many cases occurred during the fourth week after the decrease began?

21. If the average boy's height increased by a constant growth-factor between the ages of $9\frac{1}{2}$ and $14\frac{1}{2}$, find how tall he would be at $10\frac{1}{2}$, $11\frac{1}{2}$, $12\frac{1}{2}$, and $13\frac{1}{2}$. Compare the results with the table.

[You would find by multiplication that $(1{\cdot}037)^5 = 1{\cdot}2$ approximately.]

EXERCISE LII.

GROWTH PROBLEMS.

A.

1. Plot graphs for the solution of problems in which the growth-factor is (i) 1·3, (ii) 1·25, (iii) 1·1.

Five future and five past magnitudes should be plotted in each case. The numbers needed can be taken from the results in Ex. LI, Nos. 13, 14. The two graphs may be plotted on the same sheet of paper, the same vertical and horizontal scale being used for each. These graphs and the graphs drawn in the preceding lesson are those used to solve the following problems; they must therefore be executed with great care and accuracy. They are shown in fig. 50.

2. The value of the orders received daily by a certain firm is doubled in 3·1 years. What annual percentage rate of increase would lead to this result?

3. The daily output of a gold mine increased from 55 oz. to $60\frac{1}{2}$ oz. in a month. If the rate of increase remained constant, what would be the daily output (i) in two and a half months, (ii) in three and a quarter months? What was the daily output four and a half months ago?

4. Owing to the increase of motor vehicles, the number of horses in a certain town, which was 3500 in 1905, had fallen to 2173 in 1910. Assuming a constant annual rate of decrease, find the number in 1908.

5. The census of the population of the British Isles is taken every ten years. The population of a certain district in London was 8430 in 1861, and had risen to 18,510 in 1891. What was probably the population in 1885? What would you expect it to be in 1914?

6. Four tradesmen on comparing notes find that for every pound they took a year ago they have taken to-day £1 2s.,

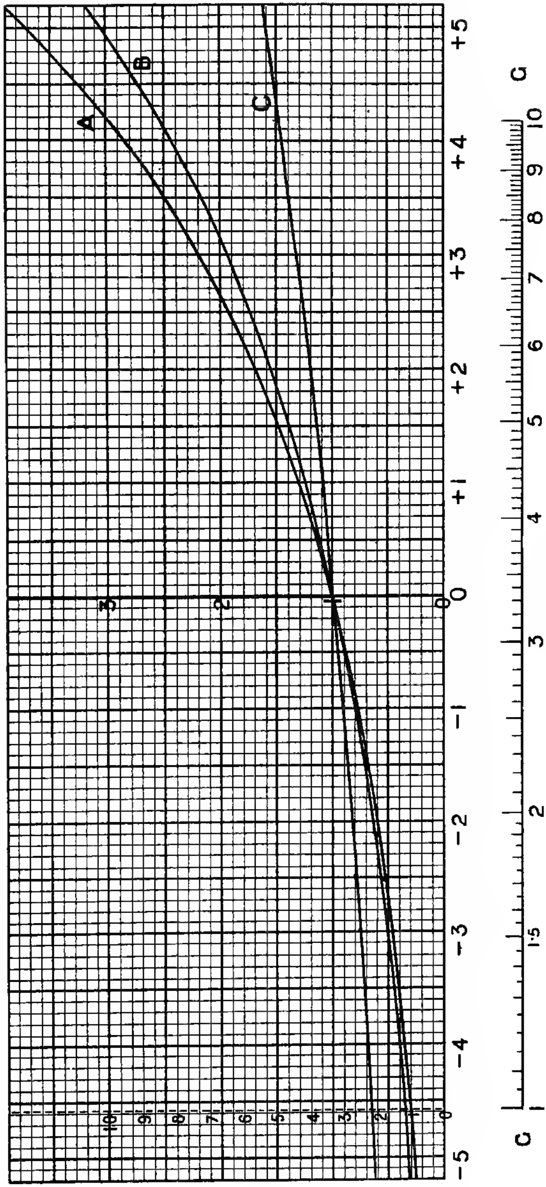


FIG. 50.

£1 6s., 16s., and £1 5s. respectively. Assuming these changes to be due to constant growth-factors, find

(i) what they may each expect their takings to be in eighteen months' time ;
(ii) what they were three months ago ;
(iii) what they were four and a quarter years ago ;
(iv) when the takings will be (or were) £2 for every £1 taken a year ago.

7. The soundings of an Admiralty surveying ship pursuing a straight course show that the depth of the sea has been and is increasing at the constant rate of 30 per cent per mile, and that at a certain place it is 100 fathoms. What will the depth be (i) 2·3 miles ; (ii) 3·6 miles ; (iii) 4·2 miles ahead? What was the depth (iv) 0·7 miles ; (v) 1·8 miles ; (vi) 4·65 miles astern of the present position of the ship?

8. What would be the depth of the sea (in the preceding problem) at a place (i) 3·3 miles ahead, (ii) 4·8 miles astern of the place where it is 240 fathoms?

9. It is found that the average score of a number of soldiers who are practising rifle shooting falls off by 10 per cent for every 100 yards that they recede from the target. This rule is found to hold good for all distances from 50 to 1000 yards. The average score at 300 yards is 24. What is it (i) at 400 yards ; (ii) at 720 yards ; (iii) at 80 yards ; (iv) at 210 yards?

10. The cost per yard of laying a railway through a mountain range increases between two given points at the rate of 25 per cent for every 10 miles. At a certain place it amounts to £2 10s. per yard. How much is it (i) 7 miles ; (ii) 14 miles ; (iii) 43 miles farther on? How much is it (iv) 17 miles ; (v) 33 miles farther back?

11. The number of people per day who visit a certain exhibition is increasing at the rate of 10 per cent per week of six days. Yesterday (which was Monday) 3620 passed the turnstiles. How many would you have expected to find (i) on the previous Thursday ; (ii) on the previous Thursday fortnight? How many would you expect to find (iii) next Tuesday ; (iv) next Friday week?

12. If the daily attendance at the exhibition began, from the Friday mentioned in No. 11 (iv), to fall off at the rate of 25 per cent per week, how many visitors might be expected (i) ten days later, (ii) fifteen days later ?

13. The population of a town has increased two and a half times in the course of three and a half years. (i) What annual growth-factor does this change represent? (ii) What increase of population per cent per annum would produce this change? (iii) Assuming a constant growth-factor, when was the population (*a*) one-quarter, (*b*) one-fifth of what it is to-day?

14. Owing to the introduction of a new machine, the manufacturers in a certain industry are now employing only 64 per cent of the men whom they employed two years ago. If this rate of decrease is maintained, how many per cent of the men at present engaged will be displaced from the industry (i) within the next eighteen months, (ii) within the next two and a half years?

15. A man bought in different parts of the same London suburb two houses for £420 each. Owing to the opening of a new railway the property rapidly rose in value, and he was able after two and three-quarter years to sell one of the houses for £860 and a year later to sell the other for £990. Find which of the two houses had been increasing most rapidly in value, and state the annual rate of increase per cent of each, assuming it to have been uniform.

B.

16. The number of persons employed by a certain firm in one of their branches has increased at a uniform rate of 10 per cent annually, since the branch was opened fourteen years ago. The number is now 195.

(i) What was the original number of the employees?
(ii) What was their number ten and a half years ago?
(iii) What was it four and a quarter years ago?
(iv) When was it exactly 100?

[Find on curve C the point whose ordinate is 1·95 and take the foot of that ordinate as the origin of the time-scale. Think out the reason why it is permissible to do this.]

17. In another branch of the same firm (opened eleven years ago) the employees number 155 and have also been increasing annually at the rate of 10 per cent.

(i) How many were they when the branch opened?
(ii) How many were they six and a half years ago?
(iii) When did they number exactly 100?

18. At the headquarters of the same firm (opened fifteen years ago) there has been the same rate of increase in the number of the staff. There are at present 636 engaged there.

(i) How many were there originally?
(ii) How many five years ago?
(iii) How many eight and a half years ago?
[Divide 636 by a number big enough to bring the quotient just within the limits of the graph.]

19. The population of an Irish village in 1841 was 1020. Every census since that date has shown a decrease of one-fifth of the population at the preceding census. What population would you expect the census of 1911 to show?

20. All rateable property (i.e. houses, tramways, etc.) in London is re-valued every five years for the purpose of determining the amount of rates to be paid. At the last quinquennial valuation in 1910 two houses were both declared to have a rateable value of £240 per annum. At the first valuation (in 1870) one of them was rated at £84. The other was not then in existence, but in 1880 was rated at £123. Assuming that the values of both houses have risen uniformly, find how much each has increased every five years.

EXERCISE LIII.

THE GUNTER SCALE.

Note.—The examples are to be solved by means of the curves and Gunter scale of fig. 50, p. 273.

A.

1. Find in each of the curves A, B, and C the abscissa of the point where the ordinate is 1·4.

2. A quantity whose present magnitude is 1·4 is increasing continuously with a growth-factor of 1·4. Write formulæ for the abscissa, x, of the ordinate which indicates its magnitude at time t (i) in curve A, (ii) in curve B, (iii) in curve C.

3. Find the magnitude of this quantity when $t = +2{\cdot}4$, by means of curve A.

4. Find the magnitude of the same quantity when $t = -0{\cdot}75$ using (i) curve B, (ii) curve C.

5. A quantity whose present magnitude is 8·4 is increasing continuously with an hourly growth-factor of 1·2. Use curve C to find :—

(i) its magnitude after $2\frac{1}{4}$ hours ;
(ii) its magnitude $1\frac{3}{4}$ hours ago ;
(iii) the time when its magnitude was 4·2 ;
(iv) the time when its magnitude will be 12·6.

6. Find the time in which the magnitude of the quantity mentioned in No. 5 will be trebled. Obtain your answer from each of two curves.

7. A quantity whose present magnitude is 200 is decreasing continuously with an annual growth-factor of 0·95. Find its magnitude (i) in three years from now, (ii) four years ago, using curve C.

8. The total length of a bean plant (fig. 2, p. 5) is to-day (Monday) 3 inches and has been increasing for some time

with a constant weekly growth-factor of 1·5. Supposing the rate of growth to remain constant, how long will it be at the same hour (i) to-morrow, (ii) on Thursday? (iii) How long was it at the same hour on Saturday last?

9. Find (i) the seventh root of 3·62; (ii) the fifth root of 2·6; (iii) the cube root of 0·7.

10. Find (i) the cube root of the fifth power of 1·3 using curve A; (ii) the fifth power of the cube root of the same number, using the same curve. Why must the results be the same?

11. Find the seventh root of the cube of 1·4 using curve B. How can you find the same by means of curve C?

12. In a certain growth-curve the ordinate whose height is a certain number, n, has an abscissa x. Write down the abscissa of the ordinates whose heights are:—

(i) the 8th root of the 5th power of n;
(ii) the 5th power of the 8th root of n;
(iii) the 6th root of the 11th power of $1/n$;
(iv) the pth power of the qth root of n;
(v) the qth root of the pth power of $1/n^3$;
(vi) the pth root of the qth power of n^a.

B.

13. By means of the Gunter scale (GG, fig. 50, p. 273) and a centimetre rule find the values of:—

(i) $(2{\cdot}2)^2$; (ii) $(2{\cdot}7)^2$; (iii) $(1{\cdot}3)^6$; (iv) $\sqrt{6{\cdot}8}$;
(v) $\sqrt{8{\cdot}1}$; (vi) $\sqrt[6]{8{\cdot}1}$; (vii) $\sqrt[3]{4{\cdot}8}$; (viii) $\sqrt[4]{9{\cdot}57}$;
(ix) $\sqrt[3]{(2{\cdot}4)^2}$; (x) $(\sqrt[3]{2{\cdot}4})^2$ (xi) $\sqrt[6]{(8{\cdot}1)^5}$; (xii) $(\sqrt[6]{8{\cdot}1})$;
(xiii) $(\sqrt[7]{3{\cdot}3})^5$; (xiv) $\sqrt[9]{(4{\cdot}8)^7}$; (xv) $\sqrt[4]{(1{\cdot}5)^5}$.

14. By means of the Gunter scale and an inch rule find the values of:—

(i) $(1{\cdot}5)^3$; (ii) $\sqrt{5{\cdot}9}$; (iii) $\sqrt[5]{8{\cdot}2}$;
(iv) $\sqrt[4]{3{\cdot}5}$; (v) $\sqrt[3]{(3{\cdot}8)^4}$; (vi) $\sqrt[3]{(1{\cdot}9)^{10}}$.

15. Take a narrow strip of paper rather more than three times as long as GG. Mark its length into three sections each equal to GG, leaving a margin at each end. Graduate the middle section as a Gunter scale, transferring from GG the graduations for the units only and leaving the positions of fractional graduations unmarked. By means of GG continue the graduation of the scale to the right, marking the positions of the tens (20, 30, . . . 100) only. Also continue

the graduation to the left, marking the positions of the tenths (0·9, 0·8, . . . 0·1) only. All the graduations should be marked with fine lines in ink.

16. On the strip of No. 15 mark lightly *in pencil* the graduation 1·85. Lay the strip across a sheet of squared paper so that the graduations "1" and "1·85" are on parallel lines of the sheet 1 cm. (or 1 half-inch) apart. Mark on the strip the positions of the graduations which have the following values :—

(i) $(1{\cdot}85)^6$; (ii) $\sqrt{(1{\cdot}85)^7}$;

(iii) the magnitude after 5·7 units of time of a quantity whose present magnitude is unity, the growth-factor being 1·85 ;

(iv) the magnitude of the same quantity 3·2 units of time before the present moment.

Read off the numerical value of these magnitudes by means of the Gunter scale on p. 273. Rub out the pencilled marks when you have finished the example.

17. Find by similar methods the value of (i) $(2{\cdot}3)^5$; (ii) $(2{\cdot}3)^{-2}$; (iii) $\sqrt[4]{(2{\cdot}3)^5}$; (iv) $\sqrt[2]{(2{\cdot}3)^7}$.

18. Lay the strip across the squared paper in such a way that the graduations "1" and "80" lie on parallel lines 12 cms. (or half-inches) apart. Mark the positions of the graduations which have the following values. Read the values off by means of the Gunter scale on p. 273:—

(i) $\sqrt[6]{80}$; (ii) $\sqrt[12]{80}$; (iii) $\sqrt[12]{(80)^7}$; (iv) $\{\sqrt[12]{(80)}\}^7$.

19. A quantity is increasing continuously with a constant growth-factor. In 10 years its magnitude increases 60 times. Find (i) the annual growth-factor ; (ii) the relative magnitude at the end of 3·6 years ; (iii) the same at the end of 7·4 years ; (iv) the same 2·5 years ago.

20. The magnitude of a quantity increases 45 times in 8·6 years. Assuming the increase to be due to a constant growth-factor, find how much the magnitude increases in 5·8 years.

21. Cut a strip of paper rather more than twice as long as the graduated base line in fig. 50, p. 273. Place the middle of the strip at the origin and mark along the strip the positions of the ordinates of curve A whose heights are 1, 2, 3, 4, and 0·9, 0·8, 0·7, . . . 0·4. Complete the graduation of the right-hand section of the strip in units from 5 to 10 and of the left-hand section in tenths from 0·3 to 0·1.

22. Use the method of No. 18 to mark on the Gunter scale of No. 21 the positions of the graduations whose values are (i) $\sqrt[4]{10}$; (ii) $\sqrt[3]{0{\cdot}1}$. Find these values numerically by means of the ordinates of curve A.

23. Write a full explanation of the method which you used to complete the graduation of your Gunter scale in No. 21.

24. The graduation "r" is at the distance x from the graduation "1" on a certain Gunter scale. Explain carefully why you expect the graduation distant xt from the graduation "1" to give the magnitude of a unit which varies continuously with a constant growth-factor r. (Consider the case when t is fractional as well as when it is integral.)

EXERCISE LIV.

LOGARITHMS AND ANTILOGARITHMS.

Note.—The references are to the growth-curves and Gunter scale of fig. 50, p. 273.

A.

1. Find by means of curve A the value of (i) $2{\cdot}2 \times 1{\cdot}6$; (ii) $2{\cdot}2 \div 1{\cdot}6$; (iii) $1{\cdot}6 \div 2{\cdot}2$; (iv) $1{\cdot}6 \div (2{\cdot}2)^2$.

2. Find by means of curve B the values of (i) $1{\cdot}5 \div 1{\cdot}2$; (ii) $1{\cdot}5 \div (1{\cdot}2)^2$; (iii) $(\sqrt{1{\cdot}5}) \div 1{\cdot}2$.

3. Answer No. 2 by means of curve C, comparing the results with those obtained by using curve B.

4. Obtain by means of either the Gunter scale or a slide rule the values of (i) $2{\cdot}7 \times 3{\cdot}4$; (ii) $3{\cdot}4 \div 2{\cdot}7$; (iii) $9{\cdot}2 \div 6{\cdot}8$; (iv) $3{\cdot}05 \times 2{\cdot}4$.

5 Obtain by the same means the value of (i) 27×340; (ii) $0{\cdot}36 \div 2800$; (iii) $0{\cdot}027 \times 0{\cdot}00035$; (iv) $4800 \div 37{\cdot}5$.

6. Obtain by means of the Gunter scale or (preferably) by a slide rule the value of (i) $4{\cdot}7 \div 3{\cdot}2 \times 2{\cdot}8$; (ii) $9{\cdot}2 \div 7{\cdot}3 \times 3{\cdot}4$; (iii) $9{\cdot}2 \times 3{\cdot}4 \div 7{\cdot}3$.

7. State in words the simplest rule for finding by the slide rule the result of arithmetical operations like those of No. 6.

8. Find the value of

$$\text{(i) } 4{\cdot}5 \times \frac{2{\cdot}8}{3{\cdot}4}; \quad \text{(ii) } 2{\cdot}9 \times \frac{3{\cdot}2}{1{\cdot}7}; \quad \text{(iii) } 2{\cdot}3 \div \frac{1{\cdot}9}{4{\cdot}7};$$

$$\text{(iv) } 4{\cdot}75 \div \frac{3{\cdot}9}{5{\cdot}4}; \quad \text{(v) } \frac{2{\cdot}8 \times 6{\cdot}2}{4{\cdot}7}; \quad \text{(vi) } 3{\cdot}1 \times \frac{7{\cdot}2}{4{\cdot}7}.$$

9. Find the value of the following by a single operation with the slide rule (or Gunter scale and paper strip). [Note that the scale must be supposed to be continued to the left of the first graduation.]

$$\text{(i) } \frac{1{\cdot}9 \times 2{\cdot}7}{3{\cdot}5}; \quad \text{(ii) } \frac{4{\cdot}3 \times 2{\cdot}6}{5{\cdot}9}; \quad \text{(iii) } \frac{8{\cdot}2 \times 7{\cdot}3}{9{\cdot}1}.$$

10. Find the value of :—

(i) $\frac{370 \times 0{\cdot}028}{2{\cdot}3}$; (ii) $\frac{425 \times 340}{290}$; (iii) $\frac{27{\cdot}5 \times 0{\cdot}082}{0{\cdot}76}$;
(iv) $\frac{2400 \times 37{\cdot}5}{0{\cdot}58}$; (v) $\frac{76 \times 505}{89}$.

11. Find the value of the following quotients : (i) $2{\cdot}7 \div 6{\cdot}4$; (ii) $3{\cdot}4 \div 7{\cdot}7$; (iii) $6{\cdot}1 \div 9{\cdot}2$; (iv) $370 \div 52$; (v) $0{\cdot}078 \div 0{\cdot}95$.

12. Find the value of the following products : (i) $3{\cdot}4 \times 4{\cdot}3$; (ii) $7{\cdot}7 \times 6{\cdot}6$; (iii) 230×72; (iv) $0{\cdot}0058 \times 0{\cdot}063$; (v) 7900×81.

B.

Note.—The work in Nos. 13 and 14 may be divided among the members of the class.

13. By means of curve B construct a table of logarithms to base $1{\cdot}25$ for numbers at intervals of $0{\cdot}1$ from $0{\cdot}5$ to $2{\cdot}0$.

14. Construct a table of antilogarithms to base $1{\cdot}25$ for logarithms at intervals of $0{\cdot}2$ from -3 to $+3$.

15. By means of the method of proportional parts calculate from your table (i) $\log_a 0{\cdot}65$; (ii) $\log_a 1{\cdot}23$; (iii) $\log_a 1{\cdot}72$. [$a \equiv 1{\cdot}25$.]

16. Calculate the value of (i) $\text{antilog}_a 2{\cdot}63$; (ii) $\text{antilog}_a 0{\cdot}8$; (iii) $\text{antilog}_a(-1{\cdot}54)$. [$a \equiv 1{\cdot}25$.]

17. Use your tables of logarithms and antilogarithms to calculate (i) $1{\cdot}8 \times 0{\cdot}65$; (ii) $1{\cdot}23 \div 1{\cdot}4$; (iii) $1{\cdot}72 \div 12{\cdot}3$.

18. Use your tables to calculate (i) $0{\cdot}6 \times 1{\cdot}6 \div 1{\cdot}4$; (ii) $(1{\cdot}5)^2 \div 1{\cdot}8$; (iii) $\sqrt[3]{1{\cdot}7} \times 1{\cdot}55$; (iv) $\sqrt[3]{0{\cdot}5} \div \sqrt{1{\cdot}9}$.

19. Complete the following identities and show how they follow from the properties of the growth-curve :—

(i) $\log_a \mathbf{P} + \log_a \mathbf{Q} =$; (ii) $\log_a \mathbf{P} - \log_a \mathbf{Q} =$;
(iii) $\log_a \mathbf{P}^n =$; (iv) $\log_a \sqrt[n]{\mathbf{P}} =$.

20. Complete the following identities and show how they follow from the properties of the growth-curve :—

(i) $\text{antilog}_a \mathbf{P} \times \text{antilog}_a \mathbf{Q} =$;
(ii) $\text{antilog}_a \mathbf{P} \div \text{antilog}_a \mathbf{Q} =$;
(iii) $(\text{antilog}_a \mathbf{P})_n =$;
(iv) $\sqrt[n]{(\text{antilog}_a \mathbf{P})} =$.

EXERCISE LV.

THE BASE OF LOGARITHMS.

A.

Note.—Nos. 1 to 6 are to be solved by means of a sheet of squared paper and a strip of paper marked, as each case requires, by reference to the Gunter scale on p. 273. [The work is done much more easily and accurately if for the strip of paper is substituted a Gunter scale cut from a sheet of semi-logarithm paper.]

1. Find the value of the following logarithms, all to base 1·5: (i) log 1·8; (ii) log 3·2; (iii) log 4·7; (iv) log 6·3; (v) log 7·9; (vi) log 10.

2. Find the value of the following antilogarithms, all to base 1·5: (i) antilog 0·5; (ii) antilog 2·3; (iii) antilog 3·0; (iv) antilog 4·7; (v) antilog 5·2.

3. Find the value of: (i) $\log_2 4·3$; (ii) $\log_2 7·5$; (iii) $\log_2 10$.

4. Find the value of: (i) $\text{antilog}_2\, 0·8$; (ii) $\text{antilog}_2\, 1·4$; (iii) $\text{antilog}_2\, 2·8$.

5. Given that $\log_x 4·9 = 3$, find (i) x; (ii) $\log_x 6·5$; (iii) $\text{antilog}_x\, 4$.

6. Given that $\text{antilog}_x\, 1·75 = 4$, find (i) x; (ii) $\text{antilog}_x\, 2·3$; (iii) $\log_x 10$.

B.

Note.—To multiply 2·357948 by 1·1 the work should be set down thus :—

$$\begin{array}{r} 2·357948 \\ ·2357948 \\ \hline 2·5937428 \\ \hline \end{array}$$

7. A Gunter scale is to be constructed in which graduations 1 cm. apart are to have a ratio of 1·1. Calculate the graduations at the points which are 1, 2, 3, . . . 10 cms. from the beginning of the scale.

8. Enter these graduations in pencil upon a strip of centimetre squared paper.[1] Mark, first in pencil and afterwards in ink, the positions of the graduations 1·1, 1·2, 1·3, . . . 2·6.

9. What would be the value of the graduation (i) 15 cms.; (ii) 16 cms.; (iii) 20 cms.; (iv) 21 cms. from the beginning of the scale? [Answers to three decimal places.]

10. A table of logarithms is to be constructed in which log 1·1 = 0·2. What will be the base of these logarithms? [Use the results of No. 7.]

11. What will be the value in this table of (i) log 1·949; (ii) log 2·358; (iii) antilog 0·8; (iv) antilog 1·6; (v) antilog 3·2?

12. Using the same base find by the method of proportional parts the value of (i) log 1·5; (ii) log 2; (iii) log 6·5.

13. A table of logarithms is to be constructed in which log 1·1 = 0·1. What is the base?

14. Find the value, with this number as base, of (i) log 1·4641; (ii) log 2·3579; (iii) log 2; (iv) antilog 0·3; (v) antilog 2·1.

Note.—Since $0{\cdot}9 = 1 - \frac{1}{10}$ the easiest way to find $0{\cdot}59049 \times 0{\cdot}9$ is as follows:—

$$\begin{array}{r} 0{\cdot}59049 \\ {\cdot}059049 \\ \hline 0{\cdot}531441 \\ \hline \end{array}$$

15. A table of logarithms is to be constructed in which, while log 1 = 0, log 0·9 = 0·25. What is the base?

16. Calculate enough results to carry this table as far as antilog 2·5.

17. Find the value in this table of (i) log 0·4783; (ii) log 0·729; (iii) log 0·4.

18. Find the value in this table of (i) antilog 1·25; (ii) antilog 2; (iii) antilog 0·6.

19. A table of logarithms is constructed in which the base

[1] If this is not available use inch squared paper and read "half an inch" for "centimetre" in Nos. 1 and 9.

is 2·1436. Find (i) antilog 0·5 ; (ii) log 2·5937 ; (iii) antilog 0·625 ; (iv) log 1·1. [Use the results of No. 7.]

20. A table of logarithms is constructed in which the base is 0·59049. Find (i) log 0·729 ; (ii) antilog 1·4 ; (iii) antilog 2. [Use the results of Nos. 15 and 16.]

EXERCISE LVI.

COMMON LOGARITHMS.

A.

Note.—Nos. 1 and 3 are to be answered by means of a Gunter scale and a sheet of squared paper. It is best to lay the Gunter scale on the paper so that the divisions "1" and "10" of the former are on two verticals of the squared paper 10 inches (or centimetres) apart. The first of these verticals should be numbered "0" and the second "1". The intermediate verticals divide this unit range into 100 equal parts.

All logarithms and antilogarithms mentioned in this exercise are "common" logarithms and antilogarithms—that is, the base is 10.

1. Find to three places the value of (i) log 2; (ii) log 3·4; (iii) 1·05; (iv) 8·7; (v) 9·3.

2. From the results of No. 1 write down the logarithms of (i) 2000; (ii) 3,400,000; (iii) 0·0034; (iv) 10,500; (v) 0·105; (vi) 0·000087; (vii) 930.

3. Find to three significant *figures* (i.e. two decimal places) the value of (i) antilog 0·4; (ii) antilog 0·75; (iii) antilog 0·368; (iv) antilog 0·945; (v) antilog 0·065.

4. From the results of No. 3 write down the antilogarithms of (i) 2·4; (ii) $\bar{3}\cdot4$; (iii) $\bar{1}\cdot75$; (iv) 4·368; (v) $\bar{6}\cdot368$; (vi) 1·945; (vii) $\bar{5}\cdot945$; (viii) 2·065.

5. From the logarithms of No. 1 calculate the logarithms of (i) $\sqrt[3]{200}$; (ii) $\sqrt[5]{(340)^2}$; (iii) $\sqrt[6]{87000}$; (iv) $(\sqrt[7]{105})^4$; (v) $3\cdot4 \times 870$; (vi) $105 \div 93$; (vii) $(3400)^2 \div \sqrt{9300}$; (viii) $\sqrt[3]{(930 \div 87)}$.

6. Find what numbers are obtained by the operations indicated in No. 5 (i) to (viii).

B.

Note.—Since the mantissa of a common logarithm is always positive while the characteristic may be either positive or negative, it follows that in addition, subtraction, etc., of logarithms the characteristics and the mantissæ must often be dealt with separately. Study the following examples :—

(*a*) $\log(0{\cdot}0034 \times 870) = \log 0{\cdot}0034 + \log 870$
$= 0{\cdot}47$
$= \log 2\ 95$

$0{\cdot}0034 = 3{\cdot}4 \times 10^{-3}$
$870 = 8{\cdot}7 \times 10^{2}$
$\log 0{\cdot}0034 = \bar{3}{\cdot}531$
$\log 870 = 2{\cdot}939$
$0{\cdot}470$

The sum is $(\bar{3} + 2) + ({\cdot}531 + {\cdot}939) = \bar{1} + 1{\cdot}470$.

(*b*) $\log(870 \div 0{\cdot}0034) = \log 870 - \log 0{\cdot}0034$
$= 5{\cdot}408$
$= \log 256000$

$\log 870 = 2{\cdot}939$
$\log 0{\cdot}0034 = \bar{3}{\cdot}531$
$5{\cdot}408$
$2{\cdot}56 \times 10^{5} = 256000$

The difference is $(2 - \bar{3}) + ({\cdot}939 - {\cdot}531) = 5 + {\cdot}408$.

(*c*) $\log \frac{1}{930} = \log 1 - \log 930$
$= \bar{3}{\cdot}032$
$= \log 0{\cdot}00108$

$930 = 9{\cdot}3 \times 10^{2}$
$\log 1 = 0{\cdot}000$
$\log 930 = 2{\cdot}968$
$\bar{3}{\cdot}032$
$1{\cdot}08 \times 10^{-3} = 0{\cdot}00108$

In this case $0{\cdot}000 - 2{\cdot}968 = 1{\cdot}000 - 3{\cdot}968$

(*d*) $\log\ (0{\cdot}093)^{3} = 3 \log 0{\cdot}093$
$= \bar{4}{\cdot}904$
$= \log 0{\cdot}000802$

$0{\cdot}093 = 9{\cdot}3 \times 10^{-2}$
$3 \log 0{\cdot}093 = \bar{2}{\cdot}968 \times 3$
$= \bar{4}{\cdot}904$
$8{\cdot}02 \times 10^{-4} = 0{\cdot}000802$

Here $\bar{2}{\cdot}968 \times 3 = \bar{2} \times 3 + {\cdot}968 \times 3 = \bar{6} + 2{\cdot}904$

(*e*) $\log \sqrt[3]{(0{\cdot}093)} = \frac{1}{3}$ of $\log 0{\cdot}093$
$= \bar{1}{\cdot}656$
$= \log 0{\cdot}453$

$\log 0{\cdot}093 = \bar{3} + 1{\cdot}968$
$\frac{1}{3}$ of $\log 0{\cdot}093 = \bar{1}{\cdot}656$
$4{\cdot}53 \times 10^{-1} = 0{\cdot}453$

The negative characteristic must first be made divisible by 3.

7. Calculate the value of :—

(i) $1050 \times 0{\cdot}00034$; (ii) $0{\cdot}87 \div 930$;
(iii) $1/8700$; (iv) $1/0{\cdot}087$.

8. Calculate :—

(i) $(0·093)^4$; (ii) $\sqrt{(0·093)}$; (iii) $\sqrt[5]{0·093}$;
(iv) $(0·00087)^3 \div (0·0105)^4$; (v) $1/\sqrt[3]{(0·93)}$.

9. Calculate :—

(i) $0·105 \times 9300 \div 8·7$; (ii) $\dfrac{3400}{10·5 \times 0·0093}$;

(iii) $\dfrac{1}{34 \times 87}$; (iv) $0·34 \times \dfrac{0·0093}{0·087}$.

10. Calculate :—

(i) $\sqrt{\{(3·4)^3 \times 93\}}$; (ii) $\sqrt[3]{\left(\dfrac{1}{(0·93)^2 \times 870}\right)}$;
(iii) $\sqrt{\{1·05 \times 930 \div \sqrt{(0·087)}\}}$.

EXERCISE LVII.

THE USE OF TABLES OF LOGARITHMS.

Note.—These examples are to be solved by means of a table of four-figure or five-figure logarithms.

A.

1. Write down the logarithms of the following numbers: (i) 4·283; (ii) 42830; (iii) 0·0004283; (iv) 2035; (v) 20·35; (vi) 0·7603; (vii) 530·2; (viii) 1007; (ix) 0·001007; (x) 8; (xi) 10,000; (xii) 0·00001.

2. Write down the antilogarithms of the following numbers: (i) 0·2430; (ii) 4·2430; (iii) $\bar{2}\cdot2430$; (iv) 3·9124; (v) 0·0325; (vi) $\bar{3}\cdot0325$; (vii) $\bar{1}\cdot0047$; (viii) 0·0006; (ix) 3·0006; (x) $\bar{5}\cdot2$.

3. Use the logarithms of No. 1 to find the value of: (i) $\log(4\cdot283 \div 20\cdot35)$; (ii) $\log(20\cdot35 \times 0\cdot001007)$; (iii) $\log(530\cdot2 \div 0\cdot0004283)$; (iv) $\log(10 \div 0\cdot001007)$; (v) $\log(1 \div 10007)$; (vi) $\log\{1 \div (530\cdot2)^2\}$; (vii) $\log(0\cdot7603)^3$; (viii) $\log(0\cdot001007)^4$; (ix) $\log\sqrt[5]{42830}$; (x) $\log\sqrt[5]{0\cdot0004283}$; (xi) $\log\sqrt[3]{0\cdot7603}$; (xii) $\log\sqrt[4]{0\cdot001007}$. [Leave a line blank below each result.]

4. Complete the calculations of No. 3 by filling in the antilogarithm of each result.

5. Find to four significant figures the value of:—

(i) $\dfrac{3\cdot28 \times 724}{23\cdot08}$; (ii) $\dfrac{0\cdot4036}{2701 \times 43}$;

(iii) $\sqrt{\left(\dfrac{7\cdot81 \times 58\cdot32 \times 0\cdot9003}{428\cdot5 \times 3\cdot027}\right)}$; (iv) $\sqrt[5]{\left\{\dfrac{14\cdot3 \times (0\cdot265)^3}{(7\cdot8)^2}\right\}}$;

(v) $\dfrac{1}{\{6\cdot237 \times \sqrt{0\cdot3856}\}^3}$.

B.

Note.—Assume $\pi = 3{\cdot}1416$, $\log \pi = 0{\cdot}49715$.

6. Use the formula $V = \frac{4}{3}\pi r^3$ to find (i) the volume of a sphere of radius 247·6 cms.; (ii) the radius of a sphere whose volume is 48·2 cubic feet.

7. Calculate (i) the amount, (ii) the present value of £340 for 16 years, allowing 4 per cent per annum compound interest.

8. How many years will it take for £1760 to amount to £3179 2s. at 3 per cent per annum compound interest?

9. The present value of a sum of £1510 16s. due in twelve years' time is stated to be £890. What is the rate of interest?

10. Find (to four significant figures) (i) the value of $(2{\cdot}7)^{20}$; (ii) the sum of 20 terms of a G.P. of which the first term is 1·3 and the common ratio 2·7.

11. Find (i) the sum of the first 30 terms of the series $1 + 0{\cdot}8 + 0{\cdot}64 + \ldots$; (ii) the sum of the second 30 terms of the same series; (iii) the limiting value of the sum of all the remaining terms of the series.

12. What percentage of the maximum sum of terms of the series $20 + 17 + 14{\cdot}45 + \ldots$ is included in (i) the first 25 terms; (ii) the first 50 terms?

13. A whole number is composed of $n + 1$ digits. What is the characteristic of its logarithm?

14. A certain number is a decimal fraction less than unity. There are $n - 1$ noughts between the decimal point and the first significant figure. What is the characteristic of its logarithm?

15. How many digits are there in (i) 2^{100}; (ii) $(37)^{500}$; (iii) $(132{\cdot}73)^{15}$?

16. How many noughts are there before the significant figures in (i) $(\frac{1}{2})^{100}$; (ii) $(\frac{1}{37})^{500}$?

17. Which is the first power of 2 whose value is above 1000?

18. Show that if $P^n \gtreqless Q$ then $1/Q \gtreqless 1/P^n$.

19. Which is the first power of $\frac{1}{2}$ whose value is below 0·001?

20. How many terms of the series $1 + 3 + 9 + 27 + \ldots$ lie between 10^4 and 10^5?

How many terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ lie between 10^{-4} and 10^{-5}?

21. How many terms of the series $1 + \frac{2}{7} + \frac{4}{49} + \ldots$ lie between 10^{-3} and 10^{-4}?

22. The first of a series of n terms in G.P. is a and the common ratio is r. Show that $r^n = S(r - 1)/a + 1$.

23. What is the smallest number of terms of the series $1 + 4{\cdot}2 + 17{\cdot}64 + \ldots$ whose sum exceeds 100,000? [Use No. 22.]

24. What is the smallest number of terms of the series $1 + 0{\cdot}9 + 0{\cdot}81 + \ldots$ whose sum exceeds 8?

C.

25. I want to buy a certain house. Instead of paying for it all at once I arrange to pay £110 for it at the end of each year for seven years. Assuming that the seller expects to receive interest at 5 per cent per annum, what is the price of the house?

26. I borrow a sum of £1250 from a Building Society on the security of a house. It is to be paid back in twenty-five half-yearly instalments, 2 per cent interest being paid per half-year. Find the amount of the half-yearly instalment.

27. A man sets aside out of his salary £20 every quarter. He deposits it in a financial institution which allows him compound interest at the rate of 4 per cent per annum—that is $\frac{1}{4}$ of 4 per cent per quarter—paid quarterly. What is the value of his investment at the end of seven years?

28. Calculate (i) the cost of an annuity of £100 a year paid half-yearly for six years, interest being reckoned at 3 per cent; (ii) the sum to which the annuity would amount by the end of the period if the instalments, when paid, were invested at 3 per cent compound interest paid half-yearly.

EXERCISE LVIII.

THE LOGARITHMIC AND ANTILOGARITHMIC FUNCTIONS.

A.

1. Draw two parallel and horizontal straight lines, and graduate them both ways from zero to ± 10. Imagine a point P to traverse the lower line from left to right, its distance at any moment from the zero point being called d. Let another point p move simultaneously along the upper line so that its position always marks the value of $\text{antilog}_a d$. Letter successive positions of the former point P_1, P_2, P_3, etc., and the corresponding positions of the latter p_1, p_2, p_3, etc. Choose the base yourself, but let it be greater than unity.

2. Below the lower scale in No. 1 draw a third graduated line. Mark on it the positions P_1', P_2', P_3', . . . which correspond to the positions marked P_1, P_2, etc., and p_1, p_2, etc., in the case when the base is a number less than unity.

3. What peculiarities in (i) the logarithmic function, (ii) the antilogarithmic function, are illustrated by the movements of the three points?

4. Explain clearly why it is (i) permissible, (ii) useful, to write

$$y = a^x \text{ instead of } y = \text{antilog}_a x.$$

5. Given that $y = (1{\cdot}3)^x$ find (by means of curve A, p. 273) the values of y when (i) $x = +1{\cdot}7$, (ii) $x = +3{\cdot}4$, (iii) $x = -2{\cdot}35$.

6. Write down the values of (i) $(1{\cdot}25)^{-1\cdot 3}$, (ii) $(0{\cdot}8)^{2\cdot 7}$, (iii) $(1{\cdot}1)^{-3\cdot 15}$, using the curves of fig. 50.

Note.—Let there be a number h and two other numbers a and b such that $h^p = a$ and $h^q = b$, p and q being integers. Also let $k = 1/p$ and $\mathbf{k} = 1/q$. Write down first the endless geometric sequence

$$\ldots \quad h^{-3},\ h^{-2},\ h^{-1},\ 1,\ h,\ h^2,\ \ldots,\ \underset{[\mathbf{a}]}{h^p},\ \ldots,\ \underset{[\mathbf{b}]}{h^q},\ \ldots \quad (A)$$

and underneath it the two endless arithmetical sequences

$$\ldots -3k,\ -2k,\ -k,\ 0,\ k,\ 2k,\ \ldots,\ \underset{[\mathrm{I}]}{pk},\ \ldots,\ qk,\ \ldots \quad (B)$$

$$\ldots -3\mathbf{k},\ -2\mathbf{k},\ -\mathbf{k},\ 0,\ \mathbf{k},\ 2\mathbf{k},\ \ldots,\ p\mathbf{k},\ \ldots,\ \underset{[\mathrm{I}]}{q\mathbf{k}},\ \ldots \quad (C)$$

Then, by the definition of logarithms, the terms of (B) are the logarithms of the corresponding terms of (A) to the base a, while those of (C) are logarithms to base b. Let $x = h^r$ be one of the numbers in (A), then we have

$$\log_a x = rk \text{ and } \log_b x = r\mathbf{k}.$$

Suppose, now, that given $\log_a x$ we want to find $\log_b x$ or *vice versa*. We have

$$\begin{aligned}\log_b x &= r\mathbf{k}\\ &= rk \times \frac{p\mathbf{k}}{pk}\\ &= \log_a x \times \frac{\log_b a}{1}\\ &= \log_a x \,.\, \log_b a.\end{aligned}$$

7. Prove, by a modification of this argument, that

$$\log_b x = \log_a x/\log_a b.$$

8. Find to base 10 the logarithms of the numbers whose logarithms are (i) $+\ 2{\cdot}6$ to base 2, (ii) $-\ 4{\cdot}8$ to base 5, (iii) $-\ 17{\cdot}6$ to base $0{\cdot}46$. Find also the numbers themselves.

9. Prove that if $y = a^x$ then $\log_b y = x \log_b a$.

10. Find by means of a table of common logarithms the value of: (i) $(10)^{2\cdot36}$, (ii) $(10)^{-3\cdot72}$, (iii) $(100)^{-2\cdot75}$, (iv) $(27)^{2\cdot3}$, (v) $(82{\cdot}1)^{-0\cdot46}$, (vi) $(0{\cdot}034)^{-6\cdot2}$.

11. Let $y' = \log_b x$, $y = \log_a x$, and $c = \log_b a$. Then we have proved that $y' = cy$.

Show that

$$b^{y'} = a^y \text{ and that } a = b^c,$$

and use the foregoing equivalence to prove that

$$(b^c)^y = b^{cy}.$$

Note.—This result is important. The argument applies equally to all kinds of values of c and y, integral and fractional, positive and negative. It proves, therefore, not only that fractional indices may be added and subtracted like integral indices but also that the rule

$$(a^m)^n = a^{mn}$$

can be used when m and n are both fractional.

12. Express each of the following as a single number with a single power-index and find its value by the curves of fig. 50: (i) $\{(1{\cdot}3)^{0{\cdot}6}\}^{1{\cdot}5}$, (ii) $\{(1{\cdot}3)^{-2{\cdot}4}\}^{-0{\cdot}5}$, (iii) $\{(1{\cdot}25)^{-3}\}^{\frac{2}{3}}$.

13. Find, by means of a table of common logarithms, the value of: (i) $\{(10)^{-3{\cdot}4}\}^{1{\cdot}5}$, (ii) $\{(2{\cdot}3)^{0{\cdot}7}\}^{3}$, (iii) $\{(0{\cdot}72)^{-3{\cdot}2}\}^{2{\cdot}3}$.

14. Prove that $x^{1/n}$ is a permissible way of expressing $\sqrt[n]{x}$, and $x^{m/n}$ a permissible way of expressing both $\sqrt[n]{(x^m)}$ and $\sqrt[n]{\{x\}^m}$. Express in the index form: (i) $\sqrt[3]{(x^2)}$, (ii) $\sqrt[5]{(x^3)}$, (iii) $\{\sqrt[7]{x}\}^5$, (iv) $\{\sqrt[4]{x}\}^5$.

B.

15. Express the following relations in the form $y = ax^m$: (i) $y^2 = 9x^3$, (ii) $y^3 = 27x^2$, (iii) $y^4 = 16x^3$, (iv) $x^{\frac{1}{2}}y^{\frac{1}{2}} = 16$. Find, whenever possible, the value of y when $x = +10$ and when $x = -10$.

16. The relation between two variables x and y is of the form

$$y = ax^m$$

Putting $\mathrm{Y} = \log y$, $\mathrm{X} = \log x$ and $c = \log a$, show that the same relation may be expressed in the form

$$\mathrm{Y} = m\mathrm{X} + c$$

the logarithms being taken to any convenient base.

17. A number of corresponding values of two variables, x and y, have been determined by measurement. When $\log y$ ($= \mathrm{Y}$) is plotted against $\log x$ ($= \mathrm{X}$) the resulting graph is a straight line cutting the Y-axis at a point c above the origin, and inclined to the X-axis at an angle whose tangent is m. What is the relation between x and y?

18. In a particular case the logarithms are taken to base 10, the line cuts the Y-axis where $\mathrm{Y} = 1{\cdot}48$ and is inclined at $13\frac{1}{2}°$ to the X-axis. Find the formula giving y in terms of x.

19. Find the relation between x and y in the following cases: (i) the line cuts the Y-axis $0{\cdot}8$ above the origin and cuts the X-axis where $\mathrm{X} = -4$; (ii) the line passes through the points $(0, +1{\cdot}85)$ and $(+2{\cdot}5, 0)$.

20. In an experiment the relation between the magnitudes of two quantities d and H is believed to follow a law of the form $d = a\mathrm{H}^m$. The following table gives a number of pairs

of values of the variables. Plot log d against log H and use the graph to determine the constants a and m :—

H	1,	2,	3,	4,	5,	6,	7,	8
d	2·2,	3·0,	3·6,	4·0,	4·5,	4·9,	5·2,	5·6

21. In an experiment the following pairs of values were determined for two quantities p and u. Plot log p against log u and use the resulting graph to determine the law which expresses the dependence of p upon u :—

u	1,	2,	3,	4,	5,	6,	7,	8
p	6·4,	3·7,	2·67,	2·15,	1·8,	1·54,	1·38,	1·23

C.

22. Show that any number x can be expressed in the form $a^{\log_a x}$, a being any positive number.

23. A table of logarithms is constructed in which the logarithms increase by equal steps $1/n$ while the numbers increase in a constant ratio $(1 + 1/n)$. What is the base of the system?

24. The base of the system described in No. 23 *can* be calculated by arithmetic, but the calculation will be tedious when n is large. On the other hand, it may be calculated by means of a table of logarithms constructed upon any principle. Use the table of common logarithms to calculate the base in No. 23 (i) when $n = 10$, (ii) when $n = 100$, (iii) when $n = 500$, (iv) when $n = 1000$.

[Log 1·002 = 0·0008677; log 1·001 = 0·0004341.]

25. Obtain by direct calculation the square root of 10 correct to four places of decimals. From this obtain the fourth root and then the eighth root. Taking these numbers as data, obtain correctly to three decimal places the values of $10^{1/8}$, $10^{2/8}$, $10^{3/8}$, . . . $10^{7/8}$, $10^{8/8}$. That is, obtain the values of antilog$_{10}$ 0·125, antilog$_{10}$ 0·25, etc.

26. Plot the logarithms 0·125, 0·25, etc., against the antilogarithms, using as open a scale as possible. Read off from your graph the logarithms of 1, 2, 3, . . . 10. Compare your readings with the value given in a table of common logarithms.

27. Determine from your graph the values of (i) log$_{10}$ 2·6, (ii) log$_{10}$ 84, (iii) log$_{10}$ 0·38. Compare your readings with the numbers in the table.

28. Simplify each of the following expressions :—

(i) $\left(a^{\frac{2}{3}} b^{-\frac{1}{2}}\right)^4 \div \left(a^{-3} b^{\frac{3}{2}}\right)^{-\frac{2}{3}}$.

(ii) $\left(\dfrac{a^{\frac{1}{2}}}{b^{\frac{2}{5}}}\right)^{-\frac{1}{2}} \times \left(a^2 b^{-\frac{2}{5}}\right)^{-\frac{3}{2}}$.

(iii) $\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)\left(x^{\frac{2}{3}} - y^{\frac{2}{3}}\right)$.

(iv) $\left(x^{\frac{1}{3}} - y^{\frac{1}{6}}\right)\left(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{6}} + y^{\frac{1}{3}}\right)$.

Note.—The student should observe that there are two typical ways of calculating logarithms. The first—which is the original method of Napier (*c.* 1594)—is the one described in No. 23 and illustrated in Ex. LV, Nos. 10, 13, 15. In this method the value of the base cannot be foreseen but has to be determined by calculation.

The second typical method is that of Henry Briggs who suggested to Napier (1615) the advantage of taking 10 as the base of the logarithms. In this method it is obvious that we must start from the base. Briggs' procedure is in essence that illustrated in Nos. 25, 26. That is to say, Briggs began by repeated extraction of roots, starting from 10. Having reached a number a very little above unity he built up his table by a method analogous to that of No. 26. But the number of root extractions, instead of being three was fifty-four! The logarithms corresponding to even distances in the scale of numbers were determined by proportion and not by graphic interpolation.

The tables of common logarithms in actual use are ultimately derived from those of Briggs.

EXERCISE LIX.

NOMINAL AND EFFECTIVE GROWTH-FACTORS.

A.

1. Find the effective annual rate of interest per pound and per cent when the nominal rate is 5 per cent per annum, the interest being added to the principal: (i) twice a year; (ii) every quarter; (iii) twelve times a year.

2. Is it more profitable to invest in a concern which offers interest at $2\frac{1}{2}$ per cent per annum and adds it to the principal every half year, or in one which gives interest at $2\frac{1}{4}$ per cent per annum and adds it to the principal every quarter?

3. A and B invest £1000 each, A in the former, B in the latter of the concerns mentioned in No. 2. What is the difference between the values of their investments after three years?

4. Interest at the nominal rate of j per £1 per annum, paid p times a year, is equivalent to an effective rate of i per £1 per annum. Show that

$$j = p\{(1 + i)^{1/p} - 1\}.$$

5. Use the formula of No. 4 to find the nominal rate equivalent to an effective rate of 21 per cent per annum, the interest being converted into principal every half-year.

6. £1000 is invested in a concern in which the interest earned is added to the principal every quarter. In five years the principal has amounted to £1346 18s. Find the rate of interest allowed per annum.

7. Write a formula for the present value (V) of a sum of money (P) due in t years, interest being reckoned at i per £1 per annum convertible p times a year.

8. Show that the cost of an annuity of a per annum, pay-

able in equal instalments p times a year for n years, interest being reckoned at j per £1 per annum, is given by the formula

$$P = a\{1 - (1 + j/p)^{-p}\}/j.$$

B.

Note.—Assume $e = 2{\cdot}718$ and $\log e = 0{\cdot}4343$.

9. Interest at a nominal rate of (i) $3\frac{1}{2}$ per cent per annum, (ii) 5 per cent per annum is converted into principal every moment. Find the equivalent effective rate in each case.

10. Interest at the nominal rate j per £1 per annum is converted every moment into principal and is equivalent to an effective annual rate of j per £1. Show that

$$j = 2{\cdot}3 \log (1 + i).$$

11. What nominal rate of interest is equivalent to an effective rate of 5·12 per cent per annum, when the interest is added to the principal as fast as it is earned?

12. Write a formula for the cost of an annuity payable continuously, the total amount payable in a year being £a. Reckon interest at the nominal rate of i per £1 per annum. [See No. 8.]

13. Interest at the nominal rate of 10 per cent per annum is paid in four different cases: (i) half-yearly; (ii) quarterly; (iii) monthly; (iv) weekly. Calculate the effective annual interest per cent in each case, using the following seven-figured logarithms:—

	Number.	*Logarithm.*
	1·001923	0·0008346
	1·008333	0·0036043
	1·025000	0·0107239
	1·050000	0·0211893
	1·102500	0·0423786
	1·103813	0·0428956
	1·104718	0·0432516
	1·105093	0·0433992
	1·105170	0·0434295
[e =]	2·718282	0·4342945

14. Draw a rectangle about one centimetre wide and of a convenient length (as great as the paper allows). Mark off areas to represent the respective values of the effective interest calculated in No. 13.

15. A man invests a sum of money at compound interest, the interest earned being added to the principal every week.

Show that if the interest were converted continuously he would possess at the end of a year only about 2d. more for every £100 invested.

C.

16. State and prove a geometrical construction for determining graphically the value of $(1 + a/n)^n$ for given values of a and n, a and n being positive numbers.

17. Show by means of a figure that the construction also applies when a is negative, so long as $|a| < 1$.

18. Prove geometrically that $(1 + a/n)^n$ approaches a definite value as n increases endlessly, a and n being both positive.

19. Prove geometrically that $(1 + a/n)^n = \{(1 + 1/n)^n\}^a$ when a is positive and $n = \infty$.

20. State and prove a geometrical construction for finding a, given the value of $(1 + a/n)^n$, a and n being both positive.

21. State and prove a geometrical construction for finding a given the value of e^a. Assume that a is positive.

22. Given that $\log 1{\cdot}000001 = 0{\cdot}000000434294$, show how to obtain an approximate value for e.

SUPPLEMENTARY EXERCISES.

EXERCISE LX.

THE USE OF LOGARITHMS IN TRIGONOMETRY.

Note.—When Napier invented logarithms (*c.* 1594) his chief aim was to lighten the labour involved in trigonometrical calculations, and the first tables which he published were tables of logarithms of sines. The present exercise is intended to give practice in the use of such tables.

In some tables the logarithms of the trigonometrical ratios are all increased by 10 in order to remove negative characteristics. Such logarithms are called "tabular logarithms". The symbol "L" is generally used (instead of "log") to indicate a tabular logarithm. Thus log sin 40° $= \bar{1}\cdot 8081$, L sin 40° $= 9\cdot 8081$.

It is assumed in the following examples that ordinary, not tabular logarithms are used. If this is not actually the case "L" should be substituted for "log" in each question.

A.

1. Write down the logarithms of (i) sin 24°; (ii) cos 36° 30′; (iii) tan 52° 10′; (iv) sin 72° 20′; (v) cos 48° 40′; (vi) cot 15°; (vii) sec 18° 20′; (viii) cosec 54° 10′.

2. Find values of α such that (i) $\log \sin \alpha = \bar{1}\cdot 7622$; (ii) $\log \cos \alpha = \bar{1}\cdot 8365$; (iii) $\log \tan \alpha = 0\cdot 0354$; (iv) $\log \cot \alpha = \bar{2}\cdot 6101$; (v) $\log \sec \alpha = 0\cdot 0582$; (vi) $\log \operatorname{cosec} \alpha = 0\cdot 0632$.

3. Write down the logarithms of (i) sin 18° 23′; (ii) cos 18° 23′; (iii) tan 54° 36′; (iv) cot 54° 36′; (v) sec 37° 45′; (vi) cosec 37° 45′.

4. Find values of α such that (i) $\log \sin \alpha = \bar{1}\cdot 6630$; (ii) $\log \cos \alpha = \bar{1}\cdot 6630$; (iii) $\log \tan \alpha = \bar{1}\cdot 8140$; (iv) $\log \cot \alpha = 0\cdot 1836$; (v) $\log \sec \alpha = 0\cdot 0340$; (vi) $\log \operatorname{cosec} \alpha = 0\cdot 0960$.

Note.—Since no negative numbers have logarithms such

numbers must be replaced in logarithmic calculations by the corresponding positive numbers. The correct sign of the final result must be determined separately.

5. Find the values of $a \sin \alpha \cos \beta$ for each set of values of the variables given in the following table :—

	(i)	(ii)	(iii)	(iv)	(v)
a	14·8	−0·36	127	−3·281	1/38
α	27°	131°	76°	248°	307°
β	62°	25°	123°	106°	142°

6. Given that $\alpha = 22°$ and $\beta = 29°$ find the value of (i) $\log \sec \frac{1}{3}(\alpha + \beta)$; (ii) $\log \sin (2\alpha + 3\beta)$; (iii) $\log (237 \sin \alpha/\sin \beta)$; (iv) $\log \{\sin (2\alpha - \beta)/\cos^2 (\alpha + 2\beta)\}$.

B.

7. Given the angles α and β and the side a of a triangle, show that the side b can be calculated by the logarithmic formula

$$\log b = \log a + \log \sin \beta - \log \sin \alpha.$$

8. Calculate the angle γ and the sides b and c in the triangles in which the following values of α, β, and a are given :—

	α	β	a
(i)	41°	62°	107
(ii)	67° 30′	53° 12′	19·6
(iii)	115° 45′	20° 15′	243·8
(iv)	12° 17′	57° 54′	82·28

9. Use logarithmic tables to calculate the remaining parts of the triangles in which (i) A = 52°, B = 73°, and c = 427 yards; (ii) A = 98°, B = 36°, and c = 1000 feet; (iii) B = 108° 20′, C = 57° 43′, b = 1582 yards.

Note.—A formula can be expressed in logarithmic form only when it consists entirely of products, quotients, powers, or roots. Thus the formula $b = a \sin \beta/\sin \alpha$ can be turned into a logarithmic form; the formula

$$\cos \alpha = (b^2 + c^2 - a^2)/2bc$$

cannot. To obtain a logarithmic formula to replace the latter we must turn to relations of the kind suggested in Ex. XL, Nos. 20-22.

10. Show by means of the equivalence

$$\cos a = 2\cos^2\frac{a}{2} - 1$$

that
$$\cos^2\frac{a}{2} = \frac{(b+c)^2 - a^2}{4bc}$$

and hence that $\cos\frac{a}{2} = \sqrt{\frac{(a+b+c)(b+c-a)}{4bc}}$.

11. Show that if $s \equiv \frac{1}{2}(a+b+c)$ then the foregoing formula becomes

$$\cos\frac{a}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Express this formula in the logarithmic form.

12. The sides of a triangle are respectively 120, 240, and 180 feet long. Calculate the angles.

13. Demonstrate by means of the equivalence

$$\cos a = 1 - 2\sin^2\frac{a}{2}$$

the formula

$$\sin\frac{a}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

14. Deduce from the results of Nos. 11 and 13 formulæ suitable for calculating (i) $\tan\frac{a}{2}$, (ii) $\sin a$ by means of logarithms.

15. Show that the area of a triangle whose sides are given can be calculated by the formula

$$\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Note.—For calculating the angles of a triangle, given its sides, any one of the formulæ for $\cos\frac{a}{2}$, $\sin\frac{a}{2}$, and $\tan\frac{a}{2}$, will suffice. The first is the easiest to remember.

16. Calculate the angles and the area of each of the triangles whose sides are given in the following table:—

	a	b	c
(i)	47	53	82
(ii)	196	212	183
(iii)	23·62	74·8	62·04
(iv)	381·7	240·08	317·66

C.

Note.—It will be seen that the formula of No. 11 makes it an easy matter to calculate α when a, b, and c are known, but that the formula cannot be used with logarithms to calculate a when b, c, and α are known. To obtain a logarithmic formula suitable for dealing with this case we have to make use of the rather complicated relation

$$\tan\frac{\beta-\gamma}{2}=\frac{b-c}{b+c}\cdot\cot\frac{\alpha}{2}.$$

Given b, c, and α this formula enables us to calculate $\beta-\gamma$. But since $\beta+\gamma=180°-\alpha$ it is easy to deduce the values of β and γ. When these are known a can be calculated by the logarithmic formula of No. 7.

Nos. 17 to 20 show how the formula for $\tan\frac{1}{2}(\beta-\gamma)$ can be demonstrated. It is assumed that $b>c$.

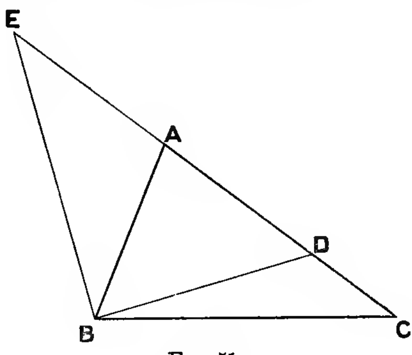


FIG. 51.

17. In the triangle ABC (fig. 51) take AD $=c$ on AC and AE $=c$ on CA produced. Join BD, BE. In the triangle BAD show that $\angle$ABD $=\angle$ADB $=90°-\frac{1}{2}\alpha$. In the triangle ABE show that $\angle$AEB $=\angle$ABE $=\frac{1}{2}\alpha$. Hence show (i) that $\angle$DBC $=\frac{1}{2}(\beta-\gamma)$ and (ii) that $\angle$EBC $=90°+\frac{1}{2}(\beta-\gamma)$. [Remember that $90°=\frac{1}{2}(\alpha+\beta+\gamma)$.]

18. In the triangle BCD prove that

$$\frac{b-c}{\sin\frac{1}{2}(\beta-\gamma)}=\frac{a}{\cos\frac{1}{2}\alpha}.$$

19. In the triangle BCE prove that

$$\frac{b + c}{\cos \frac{1}{2} (\beta - \gamma)} = \frac{a}{\sin \frac{1}{2}\alpha}.$$

20. From the last two results show that

$$\tan \frac{\beta - \gamma}{2} = \frac{b - c}{b + c} . \cot \frac{a}{2}.$$

21. The following parts of a triangle are given. Find the other parts in each case :—

(i)	$b = 24{\cdot}8$	$c = 16{\cdot}2$	$a = 42°$
(ii)	$b = 243$	$c = 328$	$\alpha = 106° \; 30'$
(iii)	$a = 167$	$b = 98{\cdot}6$	$\gamma = 50° \; 20'$
(iv)	$c = 820{\cdot}2$	$a = 604{\cdot}5$	$\beta = 122°$
(v)	$a = 120$	$c = 38$	$\beta = 8° \; 36'$

22. In Ex. XLVI, Nos. 1-5, it was shown that when the data are two sides of a triangle and the angle *opposite* to one of them the position of this side is sometimes ambiguous. Determine (by the aid of figures) in which of the following cases the ambiguity exists. Find, by the use of appropriate logarithmic formulæ, the remaining parts of each possible triangle.

(i)	$a = 143$	$b = 63$	$\beta = 38°$
(ii)	$a = 183$	$b = 206$	$\beta = 38°$
(iii)	$b = 238$	$c = 307$	$\gamma = 96°$
(iv)	$b = 11\sqrt{3}$	$c = 33$	$\beta = 30°$

EXERCISE LXI.

POLAR CO-ORDINATES.

Note.—Let P be any point whose co-ordinates are x, y. Let OP $= r$ and the angle POX $= \alpha$. Then r and α are called the **polar co-ordinates** of P. When it is necessary to distinguish x and y from the polar co-ordinates they are called the **Cartesian co-ordinates** after the great French philosopher and mathematician Descartes (1596-1650) who invented their use.

1. If x, y are the Cartesian and r, α the polar co-ordinates of a point P, show that $x = r \cos \alpha$, $y = r \sin \alpha$, and $x^2 + y^2 = r^2$.

2. When Halley's comet returned in 1909-10 it followed a path which is most conveniently described by the following **polar formula**, the origin being the sun. The symbol $d \equiv$ the comet's least distance from the sun (about 14×10^6 miles). Determine the values of r corresponding to values of α for every 15° from 0° to 360° and so plot the path of the comet:—

$$r = \frac{1{\cdot}97d}{1 + 0{\cdot}97 \cos \alpha}.$$

3. Draw the graph corresponding to the polar formula

$$r = a(1 - \cos \alpha)$$

assigning any convenient value to a. The curve is called the **cardioid.**

4. Draw the **limaçon** whose polar formula is

$$r = b - a \cos \alpha.$$

[Different members of the class should choose different values for a and b.] What does this curve become when $a = b$?

5. Draw one of the two "**three-leaved roses**" whose polar formulæ are

$$r = a \sin 3\alpha \text{ and } r = a \cos 3\alpha.$$

6. Draw one of the " **four-leaved roses** "

$$r = a \sin 2\alpha \text{ and } r = a \cos 2\alpha.$$

7. State and account for the differences between the members of each pair of " roses ".

8. Draw the **lemniscate** or figure-of-eight whose polar formula is

$$r^2 = a^2 \cos 2\alpha.$$

Explain its relation to the four-leaved rose.

9. Draw the **conchoid** whose polar formula is

$$r = a \sec \alpha + b$$
$$\text{or } r = a/\cos \alpha + b$$

10. As is well known, a properly thrown boomerang, if it does not hit its object, will return through the air towards the thrower. Experiments have been made by Mr. G. T. Walker to trace its path. The following table gives the results of one such experiment. A, B, C, . . . W, are points on the path of the boomerang; r, α are the polar co-ordinates of the points on the ground immediately below A, B, C, etc., r being measured in feet; h is the height of the boomerang in feet. (i) Draw a plan showing a " bird's-eye view " of the boomerang's flight. (ii) Join up the origin A with one of the more distant points on the plan, by a straight line. Draw a graph showing the " elevation " of the boomerang's path as viewed from a distant point along a perpendicular to that straight line.

	A	B	C	D	E	F	G	H	I	J
α	—	10°	20°	30°	40°	50°	60°	70°	80°	90°
r	0	26·5	65·5	103	124	132	134	134	134	130
h	5	10	20	26	27	30	42	51	57	55

	K	L	M	N	O	P	Q	R	S	T
α	100°	110°	115°	115°	110°	100°	90°	80°	70°	60°
r	129	115	100	72	65	59	59	62	64·5	65
h	43	33	27	19	18	18	17	17	18	19

	U	V	W
α	50°	40°	—
r	62	41	0
h	15	4	0

EXERCISE LXII.

SOME IMPORTANT TRIGONOMETRICAL IDENTITIES.

Note.—Except in taking bearings in surveying it may always be assumed that angles are measured by the anti-clockwise rotation of a line about a fixed point.

A.

1. In fig. 52 OP and PQ are both vectors of length r. The inclination of OP to the x-axis is β, that of PQ, α. What is

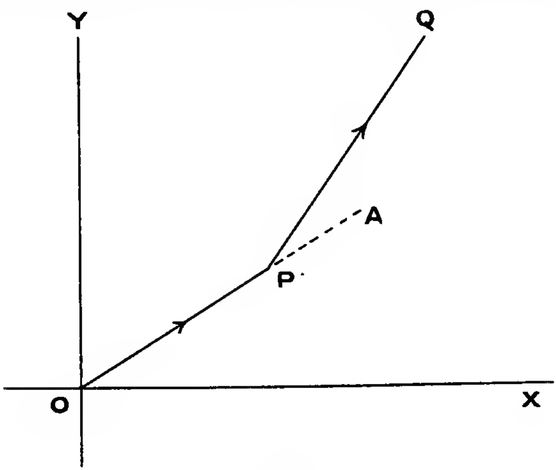


FIG. 52.

their difference of direction APQ? What is the angle AOQ? The magnitude of the resultant vector OQ? Its inclination to the x-axis?

2. Find the co-ordinates of Q (i) by projecting OP and PQ upon OX and OY; (ii) by projecting OQ upon OX and OY. Hence complete the identities

$$\cos\alpha + \cos\beta = \ . \ . \ . \ ; \ \sin\alpha + \sin\beta = \ . \ . \ .$$

3. Answer the questions of No. 1 with reference to the vectors in figs. 53 to 55, the length of each vector being r and their inclinations to the x-axis α and β. Use your

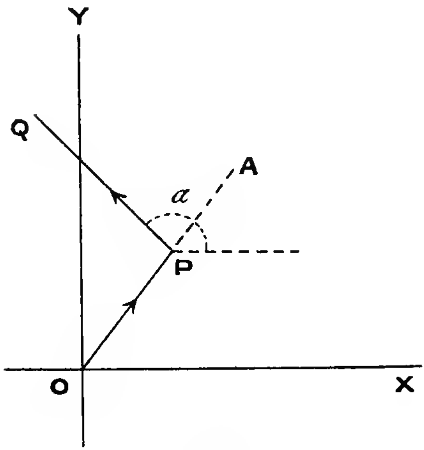


FIG. 53.

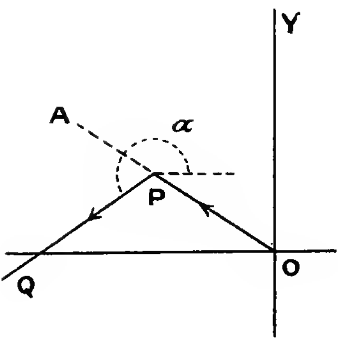


FIG. 54.

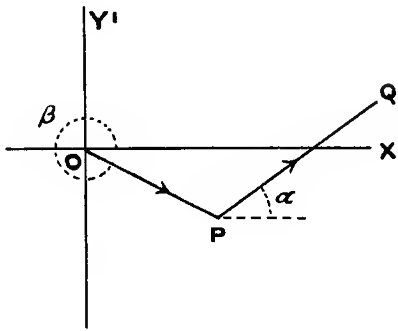


FIG. 55.

results to determine whether the identities of No. 2 hold good in all the cases shown. Are there any other possible cases? If so, draw the appropriate figures and test the identities for them.

4. In fig. 56 OQ is a resultant of two vectors, OP and PQ. The length of OQ and OP is in each case r, while their inclinations to the x-axis are respectively α and β. Show that the length of PQ is $2r \sin \frac{1}{2} (\alpha - \beta)$, and that its inclination to the x-axis is $90° + \frac{1}{2} (\alpha + \beta)$.

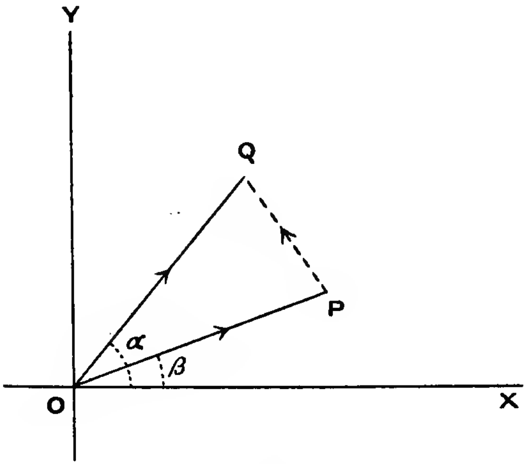


FIG. 56.

5. Find the length of the projection of PQ upon the x-axis (i) by taking the difference between the projections of OP and OQ; (ii) by projecting PQ directly, using the results of No. 4. Obtain similarly two equivalent expressions for the projection of PQ upon the y-axis. Hence complete the identities :

$$\cos \beta - \cos \alpha = \qquad ; \sin \alpha - \sin \beta = \qquad .$$

6. What products involving angles less than 90° are equivalent to :—

(i) $\cos 23° + \cos 37°$.	(ii) $\sin 23° + \sin 37°$.
(iii) $\cos 72° + \cos 48°$.	(iv) $\sin 123° + \sin 76°$.
(v) $\cos 34° - \cos 82°$.	(vi) $\sin 164° - \sin 56°$.
(vii) $\cos 342° - \cos 128°$.	(viii) $\sin 342° - \sin 128°$?

Verify any two results from the tables.

7. Express the following products as sums or differences :—

(i) $2 \cos 15^\circ \cos 32^\circ$. (ii) $2 \sin 87^\circ \sin 35^\circ$.
(iii) $2 \sin 43^\circ \cos 27^\circ$. (iv) $2 \cos 128^\circ \sin 53^\circ$.
(v) $\cos 237^\circ \sin 92^\circ$. (vi) $28 \sin 17^\circ \sin 54^\circ$.

8. Show that the following expressions are equivalent each to the tangent or co-tangent of a certain angle :—

(i) $\dfrac{\sin 43^\circ + \sin 61^\circ}{\cos 43^\circ + \cos 61^\circ}$. (ii) $\dfrac{\sin 81^\circ - \sin 24^\circ}{\cos 81^\circ - \cos 24^\circ}$.

(iii) $\dfrac{\sin 3a + \sin 5a}{\cos 3a + \cos 5a}$. (iv) $\dfrac{\sin 5a - \sin a}{\cos 5a + \cos a}$.

(v) $\dfrac{\cos 2\beta - \cos 2a}{\sin 2\beta + \sin 2a}$. (vi) $\dfrac{\cos \beta - \cos a}{\sin a - \sin \beta}$.

9. Draw a figure like fig. 56 but let $\beta = 0^\circ$. Use it to prove that

$$\cos a = 2 \cos^2 \frac{a}{2} - 1$$

and

$$\sin a = 2 \sin \frac{a}{2} \cos \frac{a}{2}.$$

Do the equivalences hold good for all values of a from 0° to 360° ?

10. Draw a figure like fig. 56 but let $\beta = 0^\circ$. What equivalences appear when PQ is projected upon the axes of x and y ? Do they agree with those of No. 9 ?

11. Find the simplest expressions by which the following may be replaced in a formula :—

(i) $\cos^2 a - \sin^2 a$. (ii) $\dfrac{\sin a}{1 + \cos a}$.

(iii) $\dfrac{\sin a}{1 - \cos a}$. (iv) $\dfrac{\cos 2a}{\cos a - \sin a}$.

(v) $\dfrac{2 \tan a}{1 + \tan^2 a}$. (vi) $\dfrac{2 \tan a}{1 - \tan^2 a}$.

(vii) $\dfrac{1 - \cos 2a}{1 + \cos 2a}$. (viii) $\dfrac{\sin a + \sin 2a}{1 + \cos a + \cos 2a}$.

12. Show that the equivalences of Nos. 9 and 10 can be deduced from those of Nos. 2 and 5 by substituting 0° for β.

B.

13. In fig. 57 OP is a vector of length r making an angle of β with a straight line OA which is inclined to the x-axis at an angle a. OQ and QP are the vectors along and at

right angles to OA by which OP may be replaced. What are the lengths of OQ and QP? Find two expressions for the abscissa of P, first by projecting OP directly on to the x-axis, secondly by projecting its component vectors OQ and QP on to the x-axis. Hence prove that

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta.$$

14. Find in a similar manner two expressions for the ordinate of P. Hence prove that

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta.$$

15. Test the general validity of these equivalences by means of figures in which OA and OP fall in various positions in the four quadrants.

16. Draw a figure similar to fig. 57 but let OP lie within the angle XOA, the angle AOP being β as before. Use this figure to prove that

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

and $$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta.$$

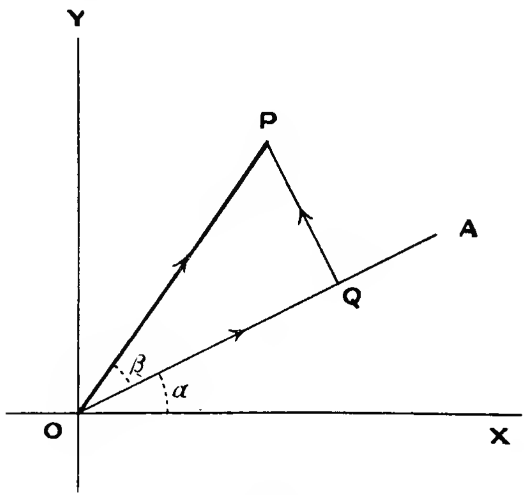


FIG. 57.

17. Use the equivalences of Nos. 13, 14, 16 to prove that

$$\begin{aligned}
\cos(\alpha+\beta)+\cos(\alpha-\beta) &= 2\cos\alpha\cos\beta\\
\cos(\alpha-\beta)-\cos(\alpha+\beta) &= 2\sin\alpha\sin\beta\\
\sin(\alpha+\beta)+\sin(\alpha-\beta) &= 2\sin\alpha\cos\beta\\
\sin(\alpha+\beta)-\sin(\alpha-\beta) &= 2\cos\alpha\sin\beta.
\end{aligned}$$

Do these identities agree with those of Nos. 2 and 5?

18. Deduce the equivalences of Nos. 13, 14, and 16 from those of Nos. 2 and 5 by substituting A for $\frac{1}{2}(\alpha+\beta)$ and B for $\frac{1}{2}(\alpha-\beta)$.

19. Use the identities of Nos. 13, 14, and 16 to find equivalences for

$$\begin{gathered}
\sin(90^\circ \pm \alpha),\ \cos(90^\circ \pm \alpha),\\
\sin(180^\circ \pm \alpha),\ \cos(180^\circ \pm \alpha),\\
\sin(270^\circ \pm \alpha),\ \cos(270^\circ \pm \alpha).
\end{gathered}$$

Do the results agree with those deduced in Ex. XLI, No. 16, from the graphs?

20. Show that $\sin(45^\circ \pm \alpha) = \dfrac{1}{\sqrt{2}}(\cos\alpha \pm \sin\alpha)$

and that $\cos(45^\circ \pm \alpha) = \dfrac{1}{\sqrt{2}}(\cos\alpha \mp \sin\alpha)$.

21. Show that $\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha} = \tan(45^\circ+\alpha)$

and that $\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} = \tan(45^\circ-\alpha)$.

Verify the identities by means of the tables, choosing any angle you please for α.

22. Show that $\dfrac{\cos\alpha+\sqrt{3}\sin\alpha}{\sqrt{3}\cos\alpha-\sin\alpha} = \tan(30^\circ+\alpha)$

and that $\dfrac{\sqrt{3}\cos\alpha-\sin\alpha}{\cos\alpha+\sqrt{3}\sin\alpha} = \tan(60^\circ-\alpha)$.

23. Demonstrate the following equivalences:—

(i) $\dfrac{1+\tan\alpha}{1-\tan\alpha} = \tan(45^\circ+\alpha)$.

(ii) $\dfrac{1-\tan\alpha}{1+\tan\alpha} = \tan(45^\circ-\alpha)$.

(iii) $\dfrac{1+\sqrt{3}\tan\alpha}{\sqrt{3}-\tan\alpha} = \tan(30^\circ+\alpha)$.

24. Establish the following identities :—

$$\text{(i)}\quad \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan(\alpha + \beta).$$

$$\text{(ii)}\quad \frac{\tan a - \tan\beta}{1 + \tan\alpha\tan\beta} = \tan(\alpha - \beta).$$

$$\text{(iii)}\quad \frac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha.$$

Show how the identities of No. 23 can be deduced from these.

EXERCISE LXIII.

THE PARABOLIC FUNCTION.

A.

1. A ball is thrown into the air. Its vertical height in feet above the point of projection is given by the formula

$$h = 72t - 16t^2,$$

the distance it has travelled horizontally from that point by the formula $d = 12t$, t being the number of seconds since it left the thrower's hand. Show by eliminating t that its path through the air is a parabola. Assuming that the ball was projected from a point 5 feet above the ground and that it falls on the roof of a house 35 feet above the level of the ground at the place where it was thrown, find the horizontal range of the ball and its time of flight. Find the horizontal range and time of flight if the ball falls into a hollow where the ground is 14 feet below the level of the ground at the place of projection. Find also the greatest height reached.

2. Suppose the ball in the previous question to be thrown straight up a hill which has a uniform slope of $\frac{1}{30}$. Taking as origin the point where the ball leaves the thrower's hand (5 feet above the ground), write the formula which describes the line of greatest slope. Find the co-ordinates of the point where the ball strikes the ground and calculate the range—that is, the distance measured along the slope from the last point to the foot of the vertical through the point of projection.

Note.—Square roots may be calculated by the approximation method or taken from tables.

3. The ball in No. 2 is now thrown straight down the hill. Calculate the range and the time of flight.

4. A ball is hit across a cricket field by a batsman. If another ball had been thrown vertically with such speed as

to be throughout its flight on the same horizontal level as the former its velocity would be given by the formula

$$v = 80 - 32t.$$

If a third ball had been rolled along the ground at a constant velocity of 20 feet/sec. it would have kept directly underneath the first ball throughout its flight. Give formulæ for the height (h) of the ball above the ground and its horizontal distance (d) from the point of projection at the different moments of its flight. Find the formula for h in terms of d. Find when and where the ball reaches its highest point and calculate its horizontal range. (It may be assumed that the ball was struck when on the ground and that the field was level.)

5. If the ball had been hit as in No. 4 but straight up a hill of slope 1/10, find (i) how far up the hill it would travel, and (ii) its time of flight.

6. Answer the questions of No. 5 in the case when the ball is hit down the hill.

7. Eliminate t from the relations $y = at - bt^2$ and $x = ct$. Find the turning value of the resulting function and the values of x and t to which it corresponds. Find the values common to the resulting relation and the relation $y = px + q$. Show that the values of t which correspond with these common values of x and y can be obtained either from $y = at - bt^2$ or from $x = ct$.

8. Two variables x and y are connected with a third variable z by the relations $x = 1 + 3z$ and $y = 1 - 2z + 4z^2$. Show that y can be expressed as a parabolic function of x and find its turning value.

9. Express y as a function of x, given that

$$y = \frac{4z^2}{1 - 3z} \text{ and that } x = 2z - 3.$$

Find the values of x and z when $y = +1$.

B.

10. Show that the parabola $y = x^2/p$ can also be described by the relation $r = p \tan \alpha \sec \alpha$.

11. Verify No. 10 by drawing the graph of $r = 4 \tan \alpha \sec \alpha$, giving α values from 0° to 360°.

(*Note.*—By r is meant the distance along OP from O towards P; account is to be taken, therefore, only of positive values of r.)

12. Move the parabola $y = x^2/p$ vertically through a distance $p/4$—downward if the parabola is "head down," upward if it is "head up". Let the x-axis now cut the curve in the points Q, Q′. Show that QQ′ $= p$.

(*Note.*—The line QQ′ is called the **latus rectum** of the parabola; the point now at the origin is called its **focus**.)

13. Show that a parabola with its focus at the origin and its latus rectum along the x-axis can be described by the polar equation

$$4r^2 - (2r \sin \alpha + p)^2 = 0.$$

Hence show that it can also be described by either of the relations

$$r = +\frac{p}{2(1 - \sin \alpha)} \text{ and } r = -\frac{p}{2(1 + \sin \alpha)}.$$

In these formulæ how will a parabola with head down be distinguished from one with head up?

14. Verify your conclusions in No. 13 by drawing the graphs of both relations on the same sheet, one half of the class putting $p = +6$ in the former and $p = -6$ in the latter, the other half reversing these substitutions.

EXERCISE LXIV.

IMPLICIT QUADRATIC FUNCTIONS (I).

A.

1. What is the relation of polar co-ordinates equivalent to

$$x^2 + y^2 = a^2$$

where a is a constant? What curve is described by this relation?

Note.—A relation such as $x^2 + y^2 = a^2$ is said to express y **implicitly** as a function of x. This means that substitution of some value for x will not suffice to give the value of y. To express the function **explicitly** we must write

$$y = \pm \sqrt{(a^2 - x^2)}.$$

Substitution of a value for x now suffices to give the value of the function.

2. The centre of the circle $x^2 + y^2 = 16$ is moved to the point $(-3, +4)$. To what implicit relation between x and y does it now correspond?

3. The centre of the circle which corresponds to the relation $x^2 + y^2 = a^2$ is moved to the point (g, f). Show that it now corresponds to the relation $x^2 + y^2 + 2gx + 2fy + c = 0$ where $c = g^2 + f^2 - a^2$.

4. What are the graphs corresponding to the following implicit functions?—

(i) $\mathbf{x^2 + y^2 - 25 = 0.}$
(ii) $\mathbf{9(x^2 + y^2) - 16 = 0.}$
(iii) $\mathbf{x^2 + y^2 - 4x + 6y - 12 = 0.}$
(iv) $\mathbf{x^2 + y^2 + 14x - 8y + 1 = 0.}$
(v) $\mathbf{x^2 + y^2 + 10x + 24y = 0.}$
(vi) $\mathbf{3x^2 + 3y^2 - 7x + 2 = 0.}$
(vii) $\mathbf{2x^2 + 2y^2 - 5y = 0.}$
(viii) $\mathbf{x^2 + y^2 - 2ax + 2by - 2(a^2 + ab + b^2) = 0.}$

(*Note.*—When the graph of a function is a circle its radius and position of its centre should be specified.)

5. Show that No. 4 (iii) is equivalent to the pair of explicit functions expressed by

$$y = -3 \pm \sqrt{\{25 - (x - 2)^2\}}.$$

Hence prove that all possible values of y are included in the range from -8 to $+2$ and those of x in the range from -3 to $+7$. Show also that the values of the function are symmetrical about -3—that is, that for every value of the function above -3 there is a corresponding value an equal distance below.

How does the graph of the function exhibit the truth of the foregoing conclusions?

6. Express No. 4 (i) and (vii) as explicit functions of x. Use the results to determine the limits within which the values of x and y must lie. Confirm your results by drawing the graphs of the functions.

7. Show (i) algebraically, (ii) by considering what the corresponding graph should be, that no values of x and y can satisfy the relation $x^2 + y^2 + 9 = 0$.

8. Apply alternately the two methods of No. 7 to find whether any values of x and y can satisfy the following relations:—

(i) $x^2 + y^2 - 4x + 6y + 20 = 0$.
(ii) $x^2 + y^2 - 7x - 18y + 100 = 0$.
(iii) $x^2 + y^2 - 10x + 4y + 29 = 0$.
(iv) $x^2 + y^2 + 8x - 12y + 52 = 0$.

9. Find the values of x and y (if there are any) which satisfy simultaneously the relation $x^2 + y^2 = 25$ and the following linear relations:—

(i) $y = 3(x + 1)$. (ii) $4x - 6y - 13 = 0$.
(iii) $3x - 4y + 25 = 0$. (iv) $9x + 7y - 49 = 0$.

Illustrate your answers by the corresponding graphs, paying particular attention to (iii).

10. Find whether the relation $x^2 + y^2 - 4x + 6y - 12 = 0$ has any values in common with the linear relations (i) $y = 3(x + 1)$ and (ii) $4x - 6y - 13 = 0$. Illustrate by graphs drawn on the same sheet as those of No. 9.

11. Show that the relations $3x - 4y + 25 = 0$ and

$$25x^2 + 25y^2 - 100x + 150y - 1524 = 0$$

have only one pair of values of x and y in common (or two identical pairs). Illustrate by a graph drawn on the same sheet as those of No. 9.

12. How could you have foretold by conside ing the functions No. 4 (i) and (iii) that the straight line

$$4x - 6y - 13 = 0$$

would pass through the intersections of their graphs? What line will pass through the intersections of the graphs of No. 4 (iii) and (iv)? Verify graphically.

13. Write down the linear relations which are satisfied by the values of x and y (if there are any) which are common to the following pairs of relations.

(i) $\mathbf{x^2 + y^2 = 4}$ **and** $\mathbf{x^2 + y^2 + 4x - 2y - 4 = 0}$.
(ii) $\mathbf{x^2 + y^2 - 6x = 0}$ **and** $\mathbf{x^2 + y^2 + 8y = 0}$.
(iii) $\mathbf{x^2 + y^2 + 4x + 2y - 4}$ **and** $\mathbf{x^2 + y^2 - 10x + 2y + 10 = 0}$.

14. Use the results of No. 13 to find the actual values of x and y (where they exist) which are common to the various pairs of relations.

B.

15. What connexion between x and y is obtained by eliminating α from the relations $x = a \cos \alpha$, $y = b \sin \alpha$?

16. Use the results of No. 15 to plot the graph of the implicit function $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. (Here $x = 3 \cos \alpha$, $y = 2 \sin \alpha$. With O as centre draw a circle of radius 3. Take any point p on its circumference; join Op and draw the ordinate pM. Then if pOX $= \alpha$, OM $= 3 \cos \alpha$ and pM $= 3 \sin \alpha$. Now if P is the point on the required graph which corresponds to this value of α its abscissa should be $3 \cos \alpha$, that is equal to OM, but its ordinate should be $2 \sin \alpha$, that is $\frac{2}{3}$ of pM. To obtain the graph, therefore, it is sufficient to draw a number of ordinates of the circle, to mark points two-thirds of the way up or down them from the x-axis, and to draw a smooth curve through these points.)

Note.—The graph of a relation of the form $x^2/a^2 + y^2/b^2 = 1$ is called an **ellipse.** It is obvious from No. 16 that its greatest length is $2a$ and its greatest breadth $2b$. These lengths measure its **major** and **minor axes.** No. 16 also shows that any chord POP′ which passes through O is bisected at O. For this reason O is called the **centre** and the chords through O **diameters** of the ellipse. The circle of radius a by the aid of which the graph is drawn is called the **auxiliary circle** and α the **eccentric angle.**

17. A piece of squared paper with axes drawn on it is fixed so that the light from the sun falls on it perpendicularly. A circular ring of radius a is held parallel to the paper and a short distance from it, the centre of the ring being directly between the centre of the sun and the origin of co-ordinates. The ring is now revolved through an angle β about the diameter whose shadow would fall on the x-axis. Show by a figure that the shadow of the ring (or the "projection" of the ring) is the ellipse $x^2/a^2 + y^2/b^2 = 1$ where $b = a \cos \beta$.

18. Use No. 17 to prove that the area of an ellipse is πab where a and b are the semi-axes. (Imagine the circular ring to be divided up into a vast number of strips perpendicular to the diameter about which it is revolved. What happens to the projected breadth, length, and area of each of these strips when the ring is turned?)

19. Hold the paper used in No. 16 so that the curve may correspond to the relation $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

20. Indicate the shape and position of the graphs of the following relations :—

(i) $x^2 + 9y^2 = 9$. (ii) $9x^2 + 25y^2 = 225$.
(iii) $4x^2 + 9y^2 = 1$. (iv) $16x^2 + y^2 = 1$.
(v) $2x^2 + 3y^2 = 6$. (vi) $7x^2 + 3y^2 = 10$.

21. Express No. 20 (ii), (iv) and (v) as explicit functions of x. Find algebraically the ranges of the possible values of x and of the function. Do they agree with the limits suggested by geometrical considerations?

22. The centre of the ellipse described by No. 20 (i) is moved to the point $(-2, +1)$. To what relation between x and y does it now correspond?

23. The ellipses described by No. 20 (v) and (vi) are moved so that their centres coincide respectively with the points $(+3, -2)$ and $(-1, -1)$. To what relations do they correspond in these positions? Does either of the curves pass through the origin?

24. Show (by reversing the method of Nos. 22 and 23) that the graphs of the following functions are ellipses. State the lengths and positions of their axes and the positions of their centres.

(i) $x^2 + 4y^2 - 2x + 24y + 33 = 0$.
(ii) $9x^2 + 4y^2 + 36x - 24y + 36 = 0$.
(iii) $2x^2 + 3y^2 - 28x + 6y + 96 = 0$.

25. Determine whether the following relations are capable of being satisfied by any values of x and y :—

(i) $2x^2 + 3y^2 + 8x - 6y + 12 = 0$.
(ii) $5x^2 + 3y^2 + 10x - 24y + 60 = 0$.

26. Find the values of x and y (if there are any) which are common to the following pairs of relations :—

(i) $4x^2 + 9y^2 = 36$ and $2x - 3y = 6$.
(ii) $4x^2 + 9y^2 = 36$ and $2x - y + 5 = 0$.
(iii) No. 38 (i) and $x + 2y + 3 = 0$.

Illustrate the results by graphs drawn on the same sheet.

EXERCISE LXV.

IMPLICIT QUADRATIC FUNCTIONS (II).

A.

Note.—Consider any point P at a distance r from the origin O. Let OP revolve in the anti-clockwise direction about O through an angle α and so carry P to a new position P′. Let the angle P′OX be β. Then the new co-ordinates of the point are $x = r \cos \beta$ and $y = r \sin \beta$ while the old ones were $r \cos (\beta - \alpha)$ and $r \sin (\beta - \alpha)$. But since

$$r \cos (\beta - \alpha) = r \cos \beta \cos \alpha + r \sin \beta \sin \alpha$$

$$\text{and } r \sin (\beta - \alpha) = r \sin \beta \cos \alpha - r \cos \beta \sin \alpha$$

the old co-ordinates can be written

$$x \cos \alpha + y \sin \alpha \text{ and } y \cos \alpha - x \sin \alpha.$$

Let P be one of the points of a graph which corresponds to a given relation between x and y. Let the whole figure be rotated anti-clockwise about O through an angle α. Then the relation between x and y which corresponds to the graph in its new position may evidently be obtained by substituting $x \cos \alpha + y \sin \alpha$ for x and $y \cos \alpha - x \sin \alpha$ for y in the original relation.

1. A graph is rotated through an angle α in the clockwise direction about O. Show that the relation to which it corresponds in its new position can be derived from the relation to which it corresponded in the former position by substituting $x \cos \alpha - y \sin \alpha$ for x and $y \cos \alpha + x \sin \alpha$ for y.

2. If the circle $x^2 + y^2 = a^2$ is rotated either way about its centre O its appearance remains unchanged. The foregoing substitutions ought, therefore, to make no difference to the relation. Show by actual substitution that this is the case.

3. The straight line $y = a$ is obviously a horizontal tan-

gent to the circle $x^2 + y^2 = a^2$. Show, by rotating the figure (anti-clockwise) through any angle α, that all tangents to the circle come under the general description

$$y = x \tan \alpha + a \sec \alpha$$

where α may be any angle from 0° to 360°.

4. Show by means of No. 3 that the line

$$y = px + a\sqrt{1 + p^2}$$

is a tangent to the circle $x^2 + y^2 = a^2$ for all values of p.

5. The rectangular hyperbola corresponding to $xy = \frac{a^2}{2}$ is rotated in the clockwise direction through 45°. Show that it now corresponds to the relation $x^2 - y^2 = a^2$. (Remember that $\sin 45° = 1/\sqrt{2} = \cos 45°$.)

6. Show that the same hyperbola when rotated through 45° in the anti-clockwise direction corresponds to the relation $x^2 - y^2 = -a^2$.

7. Express the implicit hyperbolic function $x^2 - y^2 = 9$ as an explicit function of x. Show that while the value of the function has no limits x can have no values between -3 and $+3$. Compare these results with those obtained in the case of the function $x^2 + y^2 = 9$.

8. Draw on one sheet the circle $x^2 + y^2 = 9$ and the rectangular hyperbola $x^2 - y^2 = 9$. (The easiest way to obtain the latter is to draw the asymptotes making 45° with the axes, to regard them as axes, and to draw $xy = \frac{9}{2}$ with regard to them. See No. 5.)

9. Draw any line OPP′ to cut the circle of No. 8 in P and the hyperbola in P′. Let the angle P′OX $= \alpha$. Prove that the co-ordinates of P′ are $3 \cos \alpha/\sqrt{(\cos^2 \alpha - \sin^2 \alpha)}$ and $3 \sin \alpha/\sqrt{(\cos^2 \alpha - \sin^2 \alpha)}$. Compare these with the co-ordinates of the corresponding point P upon the circle. What limits are there to the value of α?

10. Draw a series of double ordinates of the circle $x^2 + y^2 = 9$ and of the rectangular hyperbola $x^2 - y^2 = 9$. Take points on them (as in Ex. LXIV, No. 16) two-thirds up or down each ordinate from the x-axis. Draw smooth curves through the points. The curve derived from the circle is, of course, the ellipse $x^2/9 + y^2/4 = 1$. Use the expressions in No. 9 to show that the curve derived from the hyperbola is the graph of the relation $x^2/9 - y^2/4 = 1$.

11. Take points on the same ordinates (produced) four-

thirds of the way up or down them from the x-axis. Draw smooth curves through the points. To what relations do they correspond?

12. Points are taken on the ordinates of the rectangular hyperbola $x^2 - y^2 = a^2$ at distances from the x-axis which are in each case b/a of the height of the ordinate. Show that they lie on a curve which corresponds to the relation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Show that the curve has asymptotes which make angles with the x-axis whose tangents are respectively $+ b/a$ and $- b/a$.

Note.—All these curves are called hyperbolas. They differ from the rectangular hyperbola in the fact that the angles between their asymptotes is always greater or less than 90°. Remember that they are derived from the rectangular hyperbola exactly as ellipses are derived from their auxiliary circle. If the line P′O in No. 9 is produced to meet the other branch of the curve in P″ it is obvious that P′P″ is bisected at O. Chords through O are therefore **diameters** and O is the **centre** of the hyperbola. The length $2a$ (which answers to the length of the major axis of the ellipse) is called the **axis** of the hyperbola. The length $2b$ (which answers to the minor axis of the ellipse) is not really an axis of the hyperbola $x^2/a^2 - y^2/b^2 = 1$; but, as it is the axis of the **conjugate hyperbola** $x^2/a^2 - y^2/b^2 = -1$ it is called the **conjugate axis** of the former hyperbola.

13. Show that the following are hyperbolic functions. Indicate the positions of the corresponding curves; give the lengths of their axes and state the angle between their asymptotes.

(i) $x^2 - y^2 + 4x + 6y - 9 = 0$.
(ii) $x^2 - 4y^2 + 8x + 24y - 24 = 0$.
(iii) $4x^2 - 9y^2 - 8x - 36y + 4 = 0$.
(iv) $3x^2 - 2y^2 + 12x + 12y - 12 = 0$.

14. Show that values of x and y can always be found which will satisfy relations of the forms

$$\frac{(x+p)^2}{a^2} - \frac{(y+q)^2}{b^2} = 1 \text{ and } \frac{(x+p)^2}{a^2} - \frac{(y+q)^2}{b^2} = -1.$$

What limits (if any) are there to the values of x and y in each case?

15. Show that if a and b are positive a relation of the

form $ax^2 - by^2 \pm 2gx \pm 2fy \pm c = 0$ can always be thrown into one of the forms in No. 14 and hence will always be satisfied by some values of x and y.

B.

16. The parabola $y = p^2x^2$ is revolved (i) clockwise through 90°; (ii) anti-clockwise through 90°; (iii) anti-clockwise through 180°. Apply the substitution formulæ to determine the relations describing it in each of these positions. Do they agree with former results?

17. The co-ordinates of a point P are p and q. The line OP is rotated, anti-clockwise, through an angle α. Show that the new co-ordinates of P are

$$p \cos \alpha - q \sin \alpha \text{ and } q \cos \alpha + p \sin \alpha.$$

Find the new co-ordinates of a point P, originally at $(+5, -10)$ after OP has been rotated anti-clockwise through the angle whose tangent is 3/4.

18. The line joining O to the centre of the circle

$$x^2 + y^2 - 10x + 20y + 109 = 0$$

is rotated anti-clockwise through the angle whose tangent is 3/4. Find (by the substitution of the Note to No. 1) the relation corresponding to the circle in its new position. State the old and new positions of the centre of the circle. Compare the results with those of No. 17. Do they agree? Is the radius of the circle unaltered (as, of course, it should be) by the rotation?

19. Find the relation corresponding to the parabola $y = -2x^2$ after it has been (i) rotated anti-clockwise through 30° and (ii) afterwards moved parallel to itself until its vertex is at the point $(-3, +10)$.

20. Give a formula descriptive of the ellipse $2x^2 + 3y^2 = 4$ after it has been (i) turned clockwise through 45°, and (ii) moved parallel to itself until its centre is at the point $(+4, -2)$.

21. Find the implicit function of x whose graph is the hyperbola $4x^2 - 3y^2 = 2$ (i) rotated anti-clockwise through the angle whose tangent is $\frac{1}{2}$, and (ii) moved parallel to itself until its centre is at the point $(-1, -1)$.

22. Find the relation between x and y the graph of which is a rectangular hyperbola whose axis is 5 and is inclined to the x-axis at an angle whose tangent is $\frac{1}{3}$.

Note.—Nos. 16-22, taken with earlier examples, bring out the following facts about functions in which both x and y are present in the second degree (and possibly in the first degree also) :—

(i) If the graph is a circle there can be no co-efficient of xy and the co-efficients of x^2 and y^2 must be identical ;
(ii) If the graph is a parabola, an ellipse or a hyperbola, xy will be present unless the axis is parallel either to the x-axis or the y-axis ;
(iii) If the function is parabolic the three terms of the second degree will form a perfect square ;
(iv) If the function is elliptic the co-efficients of x^2 and y^2 will be unequal (unless $a = 45°$) but both of the same sign ;
(v) If the graph is a rectangular hyperbola the co-efficients of x^2 and y^2 will be equal but opposite in sign ;
(vi) If the graph is a non-rectangular hyberbola the co-efficients of x^2 and y^2 will be opposite in sign and unequal ;
(vii) Values of x and y can always be found which will satisfy a given function if it is parabolic or hyperbolic in form but not always if it is circular or elliptic in form.

23. State what you can determine by inspection about the character of the following functions :—

(i) $(3x - 4y)^2 - 7x + 8y - 11 = 0.$
(ii) $3x^2 + 3y^2 - 4x + 6y + 1 = 0.$
(iii) $3x^2 - 3y^2 - 4x + 7y + 1 = 0.$
(iv) $9x^2 + 4xy + 7y^2 - 8x + 5y + 3 = 0.$
(v) $5x^2 + y^2 + 7x - 13 = 0.$
(vi) $2x^2 - 5xy - 3y^2 + 2x - 7y + 18 = 0.$

Note.—If the term involving xy has been brought into the formula of a curve by rotating the curve anti-clockwise about the origin, it can obviously be removed by rotating it back through an equal angle.

24. Show that the relation $14x^2 - 4xy + 11y^2 = 5$ describes an ellipse which has been rotated anti-clockwise through the angle whose tangent is 2.

25. Show that the graph of the function

$$14x^2 - 36xy - y^2 - 13 = 0$$

is a hyperbola which, if rotated clockwise through the angle whose tangent is $\frac{3}{2}$, would have its axis coincident with the y-axis.

26. The rectangular hyperbola $3x^2 + 8xy - 3y^2 = 10$ is rotated backwards (i.e. clockwise) until its axis is coincident with the x-axis. Show that the angle is given by the equation

$$4(\cos^2 \alpha - \sin^2 \alpha) - 6 \sin \alpha \cos \alpha = 0.$$

Show that this equation can be written in the form

$$3 \sin 2\alpha = 4 \cos 2\alpha$$

Hence show that $\alpha = 26\frac{1}{2}°$.

27. Find the angle through which the ellipse

$$37x^2 - 18xy + 13y^2 = 10$$

must be turned back so that its major axis may coincide with the x-axis.

28. Show that the graph of the function

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will have its axis parallel to the x-axis if it is rotated clockwise through an angle α such that $\tan 2\alpha = 2h/(a - b)$. (Note that the terms $2gx$ and $2fy$ affect only the position of the centre, not the inclination of the axis.)

29. Show that the relation

$$5x^2 + 2xy + 5y^2 - 6x + 18y + 11 = 0$$

may be expressed as an explicit function of x in either of the forms

$$y = -\tfrac{1}{5}(x + 9) \pm \tfrac{1}{5}\sqrt{\{50 - 24(x - 1)^2\}}$$

or

$$y = -\tfrac{1}{5}(x + 9) \pm \tfrac{1}{5}\sqrt{(5\sqrt{2} - 2\sqrt{6} + 2\sqrt{6}x)(5\sqrt{2} + 2\sqrt{6} - 2\sqrt{6}x)}.$$

How can you deduce from these expressions (i) that the line $y = -\frac{1}{5}(x + 9)$ bisects all vertical chords of the ellipse; (ii) that it meets the curve at the points

$$(+1 + \tfrac{5}{6}\sqrt{3}, -1 - \tfrac{1}{6}\sqrt{3})$$

and $(+1 - \frac{5}{6}\sqrt{3}, -1 + \frac{1}{6}\sqrt{3})$; (iii) that the tangents at these points are vertical; (iv) that the tangents at the points $(+1, -2 + \sqrt{2})$ and $(+1, -2 - \sqrt{2})$ are parallel to the line $y = -\frac{1}{5}(x + 9)$? Use these results to draw with rough accuracy the graph of the function.

30. As we have seen, the relation $p^2x - q^2y^2 = r$ represents a hyperbola with its centre at the origin. Show that the graph of $p^2x^2 - q^2y^2 = 0$ is a pair of straight lines through the origin. What is the connexion between this pair of lines and the hyperbola?

Find, by converting it into an explicit fuuction of x, whether the relation

$$2x^2 + xy - 6y^2 + 7y - 2 = 0$$

corresponds to a hyperbola or a pair of straight lines.

EXERCISE LXVI.

MEAN POSITION.

A.

1. Fig. 58 represents a target 4 feet long at which a man has been shooting with the object of hitting the vertical line CC′. Find the mean distance of his hits from CC′ by adding the various distances (regarded as directed numbers) and dividing the resultant by the total number of shots.

Note.—The above directions for finding the mean distance could be expressed by the formula $\bar{x} = (\Sigma x)/n$ where $\bar{x} \equiv$ mean distance from CC′ and $n \equiv$ number of shots.

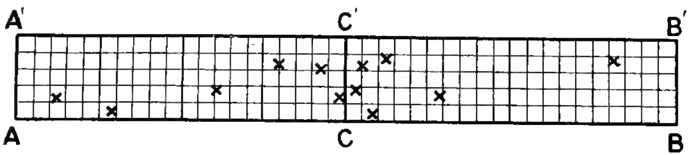


FIG. 58.

2. Let a line MM′ be drawn across the target so that its distance from CC′ is the mean distance calculated in No. 1. Show that the position of MM′ can also be obtained by calculating the mean distance of the hits from AA′.

3. What is the mean distance of the hits from MM′? From a line 1·2 feet to the left of MM′? From a point 0·6 feet to the right of CC′? (These questions can be answered without calculation.)

4. Fig. 59 shows the attempts of a marksman to hit the centre of a target 4 feet square. Find the mean distance of his hits from each of the lines YY′, XX′.

Note.—Mark in fig 59 a point whose co-ordinates are the values of $\bar{x}$ and $\bar{y}$ calculated in No. 4. This point is the

mean position of the hits. It is called the **centroid** of the points where the shots hit the target.

5. What is the distance of the centroid from the centre of the target in No. 4 ?

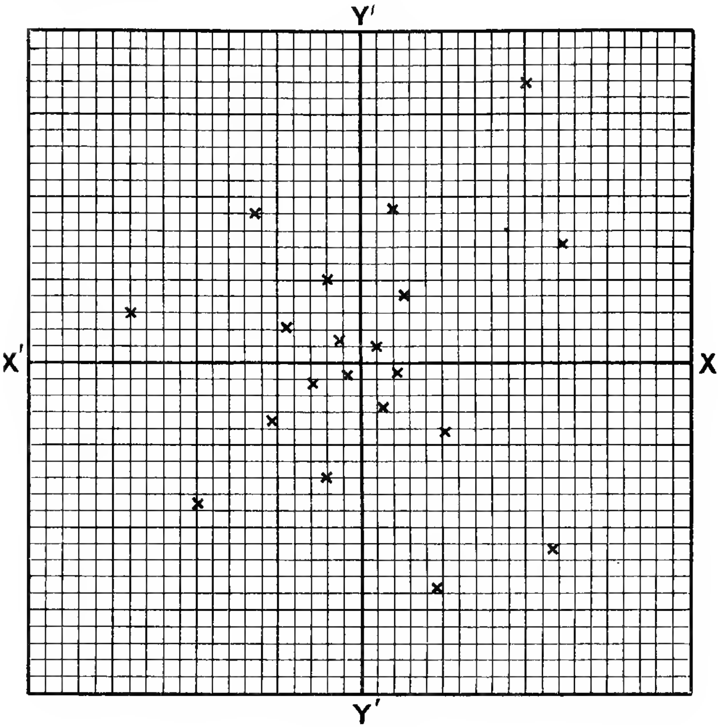


FIG. 59.

6. When another marksman shot at the same target the co-ordinates of the points at which he hit it were :—

$(-1{\cdot}2, +0{\cdot}6)$, $(+2{\cdot}1, +0{\cdot}4)$, $(0, -0{\cdot}2)$, $(-0{\cdot}7, -0{\cdot}5)$, $(+0{\cdot}8, 0)$, $(0, 0)$, $(+0{\cdot}1, -0{\cdot}3)$, $(+0{\cdot}9, +0{\cdot}6)$, $(-1{\cdot}3, +0{\cdot}4)$, $(0, 0)$, $(+0{\cdot}6, -0{\cdot}4)$, $(0, +1)$. Calculate the co-ordinates of the centroid and its distance from the centre of the target.

B.

7. Take any triangle OAB and from O draw a perpendicular OP meeting AB in P. Let OP = h and AB = kh. Imagine the surface of the triangle to consist of a very large number of very small squares arranged in rows parallel to AB. Let there be n squares in each unit of length. How many squares will there be in the row AB? How many in a row distant x from O? How many in the whole triangle?

8. Assuming that the squares are so small that they may be treated as points, show that the centroids of all the rows lie on a certain straight line.

9. On the same assumption show that the mean distance of the surface of the triangle from a line through O parallel to AB is $\frac{2}{3}h$.

10. What is (from Nos. 8 and 9) the exact position of the centroid of a triangle?

11. Find the centroid of a triangle by calculating the mean distance of the elements of its surface from the base.

12. Apply the methods of Nos. 7-10 to find the centroid of a hollow cone without a base.

13. Find the centroid of a solid cone. [The cone must be thought of as built up of a vast number of very small cubes or "elements of volume".]

14. Show that the centroid of a pyramid on a square base is three-quarters of the distance from the vertex to the centroid of the base along the line joining them.

15. A cap is cut off from a thin india-rubber or metal ball. Where is the centroid of the residue?

16. A conical shell (without a base) is fashioned from a block of material in such a way that the thickness of the shell at any point is proportional to its distance from the vertex. Where is the centroid?

17. Two sides AB, CD, of a quadrilateral are parallel and are respectively 8 inches and 35 inches long. The perpendicular distance between them is 9 inches. What is the length of a line x inches from AB? Where is the centroid?

18. On a sheet of paper outline any two figures of area A_1 and A_2. Mark two points, C_1 and C_2, to represent their centroids. Let the co-ordinates of C_1 and C_2 be respectively $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$. Let C be the centroid of all the elements of area included in the two figures, and let its co-ordinates be

$\bar{X}$, $\bar{Y}$. Show that $\bar{X} = (A_1\bar{x}_1 + A_2\bar{x}_2)/(A_1 + A_2)$ and that $\bar{Y} = (A_1\bar{y}_1 + A_2\bar{y}_2)/(A_1 + A_2)$.

19. Supposing that you know $\bar{X}$, $\bar{Y}$ and x_1, y_1, by what formulæ would you calculate $\bar{x}_2$, $\bar{y}_2$?

20. Find the centroid of a figure which consists of an isosceles triangle of height h standing on one side of a square measuring a each way.

21. Instead of standing on the side of the square the isosceles triangle has been cut out of the square (h being $< a$). Where is the centroid of the remaining area?

22. A figure is composed of two circles which touch one another externally, their radii being 5 inches and 12 inches respectively. Where is the centroid?

23. In another figure the smaller circle has been drawn so as to touch the larger internally and has then been cut out. Where is the centroid of the remaining area?

24. A portion of a cone has been removed by a section parallel to the base. The radius of the top is 10 cms., that of the base 24 cms., while the line joining their centres is 42 cms. long. Write a formula for the area of a section distant x from the top. Hence calculate the position of the centroid.

25. Find by measurement and calculation the position of the centroid in figs. 11 and 12 (p. 34).

C.

Note.—The Greek mathematician Pappus (*c.* 350 A.D.) and, in modern times, the Swiss Guldinus (*c.* 1635) demonstrated two important theorems of mensuration:—

Imagine the curve P (fig. 60) to revolve in space about the external axis YY′, so as to mark out a solid ring-like figure. Let l be the perimeter of the curve and A its area. Also let C_1 be the centroid of the *perimeter* of the curve and C_2 the centroid of its *surface*. Then we have—

surface of ring-solid $= l \times$ circumference of circle traced out by C_1 . . . (I)

volume of ring-solid $=$ A $\times$ circumference of circle traced out by C_2 . . . (II)

Nos. 26-28 give the argument leading up to I, Nos. 29-31 that leading up to II.

26. Suppose the perimeter of the curve to be divided up into a large number (n) of equal segments of length δl. Mark the mid-length of each segment by points of which P is a specimen. Let the distance of P from the axis be x, and let the distance of C_1, the centroid, be $\bar{x}$. Prove that

$$\Sigma 2\pi x = 2\pi n\bar{x}.$$

27. Write down an expression for the area of the surface traced out by the segment P on the assumption that it is a straight line of length δl. Note that by taking δl small enough this assumption may be made true to within c per cent for every segment, c being chosen as small as we please.

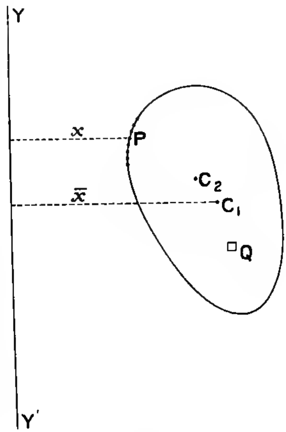


FIG. 60.

28. Hence show that

$$\text{surface of ring-solid} = 2\pi\bar{x} \,.\, l$$

to any nameable degree of accuracy.

29. Next suppose the surface of the curve to be mapped out into n squares of area δA of which Q is a specimen. Let the length of the side of the square be $2h$ and the distance of its mid-point from YY′ be x. Also let the distance from YY of C_2, the centroid of the area, be $\bar{x}$. Show as before that

$$\Sigma 2\pi x = 2\pi n\bar{x}.$$

30. Find an expression for the volume of the solid of square section traced out by the revolution of Q.

31. Hence show that

$$\text{volume of ring-solid} = 2\pi\bar{x} \,.\, A.$$

32. The cross-section of the iron of an anchor-ring is a circle of radius r. The diameter of the whole ring is 2R. Write down formulæ for finding (i) its surface A and (ii) its volume V.

33. Calculate the surface and volume of an anchor-ring in which $r = 3$ inches and $R = 8{\cdot}2$ inches. [Assume $\pi^2 = 10$.]

34. ABC is an isosceles triangle in which BC = 10 inches and AB = AC = 13 inches. A solid figure is produced by revolving the triangle about an axis 16 inches from, and

parallel to, its base. Calculate the area and volume of the solid.

35. Fig. 27 (p. 94) is identical with the cross-section of the rim of a wheel. The mean radius of the wheel is r. Write a formula for calculating its area and its volume.

36. An anchor-ring has an elliptical cross-section, the major axis being 4 inches and the minor 3 inches long. The mean radius of the ring is 6 inches. Find its weight, given that 1 cubic inch of iron weighs 0·28 lb. [See Ex. LXIV, No. 18.]

EXERCISE LXVII.

ROOT-MEAN-SQUARE DEVIATION.

A.

Note.—If a measure is required of the inaccuracy of the shooting in Ex. LXVI, No. 1, it must be noted (*a*) that a deviation (say) of 1 foot to the right of the centre of the target and a deviation of 1 foot to the left have exactly the same importance, and (*b*) that a deviation of 1 foot is more than 12 times as serious as a deviation of 1 inch—that is, the badness of a shot must not be measured simply by the distance of the point hit from the point aimed at. For both these reasons it is usual, in estimating the departure of the hits from perfect accuracy, (i) to square their actual distances from the point aimed at, (ii) to find the mean of the squares, and (iii) to take the square root of the mean. For by squaring the distances a negative distance is rendered of the same importance as a positive distance of the same magnitude, and larger deviations acquire relatively more influence upon the estimate than smaller deviations. The result is called **the square root of the mean square of the deviations**—or, more concisely, the **root-mean-square deviation.**

1. Calculate the square root of the mean square of the deviations from CC' in fig. 58.

2. In fig. 59 find the square root of the mean square of the distances of the points hit (i) from XX', (ii) from YY'.

3. Show that the mean square of the distances of a number of points from the origin is the sum of the mean squares of their distances from the axes. Apply this principle to calculate the square root of the mean square of the distances of the hits from the centre of the target in fig. 59.

4. Calculate the square root of the mean square of the distances from the centre of the hits recorded in Ex. LXVI, No. 6.

5. In an examination in arithmetic ten questions were set and the answers were marked as either "Right" or "Wrong". The class was divided into two divisions, A and B, each containing 15 pupils. The following table gives the number of pupils in each division who obtained 10, 9, 8 . . . 0 correct answers. Calculate (i) the mean number of correct answers in each division, (ii) the square root of the mean square of the deviations from perfect accuracy in each division. Which division did best? Why do you think so?

No. correct:	10	9	8	7	6	5	4	3	2	1	0
A:	1	0	1	1	5	4	0	2	0	1	0
B:	3	0	0	2	2	3	1	0	3	1	0

B.

Note.—In studying the behaviour of rotating bodies and in other problems it is often necessary to answer questions like Nos. 6-20.

6. On a line l cms. long are strung $m + 1$ equal spherical beads touching one another. Their centres are d cms. apart —that is, $l = md$. Show that the root mean square of the distances of the centres of the beads from the centre of the end one is

$$l\sqrt{\left(\frac{1}{3} + \frac{1}{6m}\right)}$$

7. What is the root-mean-square of the distances of the points of a straight line of length l from one of its ends?

8. Find the root-mean-square of the distances of the points of a line of length l from its mid-point.

9. Calculate the root-mean-square of the distances of all the points of the rectangle AB′ (fig. 58) from the line CC′.

10. Calculate the root mean square of the distances of the superficial elements of a soap bubble (i) from a tangent plane, (ii) from a plane through its centre.

11. OAB is any triangle. The perpendicular distance between O and AB is h. A number of equidistant lines are drawn across the triangle parallel to AB. Including AB and the line of zero length through O there are $m + 1$ of them altogether. Show that the root-mean-square of the distances of all the points on these lines from a line through O parallel to AB is

$$h\sqrt{\left(\frac{1}{2}+\frac{1}{2m}\right)}$$

12. What is the root-mean-square of the distances of the points of the triangle OAB from the line through O parallel to AB?

13. Find the root-mean-square of the distances of the points of the figure in Ex. LXVI, No. 17, from the side AB.

14. Repeat the last investigation substituting CD for AB. Does the answer agree with that of No. 13?

15. A figure is composed of a series of concentric circles whose radii (if the centre be regarded as one of the circles) are in direct proportion to the numbers 0, 1, 2, . . . m, the radius of the largest being r. Find the root-mean-square of the distances from the centre of all the points on the circumferences of these circles.

16. What is the root-mean-square of the distances from the centre of all points in the area of a circle of radius r?

17. Deduce from the result of No. 16 by means of the principle of No. 3 the root-mean-square of the distances of the points within a circle from one of the diameters.

18. Find the root-mean-square of the distances of the points on the circumference of a circle from one of the diameters.

19. Show that the root-mean-square of the distances of the points on the surface of a sphere from a plane through its centre is $r/\sqrt{3}$.

20. Calculate the root-mean-square of the distances of the points in fig. 14 (p. 35) from the line joining two opposite vertices, assuming $a = 2$, $b = 1$. Deduce the root-mean-square of the distances from the centre of the figure.

EXERCISE LXVIII.

THE BINOMIAL THEOREM.

Note.—If we want to know exactly how much a sum of money £P will amount to in n years at i per pound per annum, compound interest, we must use the formula $A = P(1 + i)^n$ and evaluate the second factor either by direct calculation or by the aid of logarithms. But if an approximate value will suffice it can be obtained by "expanding" $(1 + i)^n$ in accordance with Stifel's table (Ex. XXXI) and replacing $(1 + i)^n$ by as many terms of the expansion as will suffice for the degree of accuracy in view. Thus, suppose that we want to know the amount of £100 for 6 years at 3 per cent, compound interest. By Stifel's table we have

$$(1 + i)^6 = 1 + 6i + 15i^2 + 20i^3 + 15i^4 + 6i^5 + i^6$$

whence

$$\begin{aligned}(1\cdot03)^6 = (1 + 0\cdot03)^6 &= 1 + 0\cdot03 \times 6 + (0\cdot03)^2 \times 15 + \\ &\qquad (0\cdot03)^3 \times 20 + \ldots \\ &= 1 + 0\cdot18 + 0\cdot0135 + 0\cdot00054 + \ldots \\ &= 1\cdot19404 + \\ \therefore \quad A &= £1\cdot19404 \times 100 + \\ &= £119\ 8s. +\end{aligned}$$

The last figure of the approximate factor is ignored because the next term of the series might affect it. (This term is, in fact, 0·00001215—a number which affects the ultimate result to the extent of rather more than a farthing.)

This method of approximation can always be used provided (i) that $i < 1$ and (ii) that n is a positive integer. If i were > 1 the approximation would be impossible because the terms would, as a rule, increase successively instead of decreasing. If n is not a positive integer Stifel's table will not enable us to determine the coefficients of the terms.

In dealing with a long term of years Stifel's method of finding the coefficients required for the expansion of $(1 + i)^n$ is inconvenient. Fortunately **Sir Isaac Newton** discovered (about 1665) a formula by which the coefficients required for a given value of n can be determined without reference to those needed for other values. His rule is that the coefficients of the terms (including the first) are :—

$$1,\ 1 \times \frac{n}{1},\ \frac{n}{1} \times \frac{n-1}{2},\ \frac{n(n-1)}{1 \,.\, 2} \times \frac{n-2}{3},\ \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} \times \frac{n-3}{4},\ \text{etc.}$$

Thus when $n = 6$, the coefficients are 1, $1 \times \frac{6}{1}$, $6 \times \frac{5}{2}$, $15 \times \frac{4}{3}$, $20 \times \frac{3}{4}$, $15 \times \frac{2}{5}$, $6 \times \frac{1}{6}$. The law of the decreasing numerators and increasing denominators of the successive multipliers is obvious.

The products which constitute the denominators of these coefficients are called **factorials**. Thus the product $1 \,.\, 2 \,.\, 3 \,.\,.\,.\, r$ is called **factorial** r. It is generally expressed by the symbolism $\underline{|r}$ or $r\,!$

In honour of their discoverer the above expressions will be called the **Newtonian coefficients**. It is most convenient to refer to the first (which is always unity) as the "zeroth," and the others as the first, second, . . . Newtonian coefficients.

A.

1. Expand (i) $(1 + i)^5$, (ii) $(1 + i)^8$, (iii) $(1 + i)^{10}$ by means of Newton's rule and compare the results with those obtained by Stifel's table.

2. Write down the formulæ for (i) the r^{th} Newtonian coefficient (c_r), (ii) the $(r + 1)^{th}$ term of the expansion of $(1 + i)^n$.

3. Show that $c_r = n\,!/r\,!\,(n - r)\,!$ Which other coefficient is described by the same expression?

4. Find by algebraic multiplication the product of—

$$1 + ni + \frac{n(n-1)}{2\,!}i^2 + \frac{n(n-1)(n-2)}{3\,!}i^3 + \ .\ .\ .$$
$$.\ .\ . + \frac{n(n-1)\ .\ .\ .\ (n-r+1)}{r\,!}i^r + \ .\ .\ .\ ni^{n-1} + i^n$$

by $(1 + i)$, arranging the result in order of the powers of i.

Note.—No. 4 is very important for it shows that if Newton's formula holds good for the expansion of $(1 + i)^n$ it also holds

good for the expansion of $(1 + i)^{n+1}$. But we know by direct multiplication that it holds good for $(1 + i)^2$. Hence **it holds good for all positive integral values of** n.

5. Write out from memory the argument of No. 4 and the preceding note as a proof by recurrence (or "mathematical induction") that Newton's expansion of $(1 + i)^n$ is true for all positive integral values of n.

6. Use No. 3 to prove that, when n is a positive integer, (i) the expansion of $(1 + i)^n$ contains $n + 1$ terms; (ii) if n is even there is a middle term and the Newtonian coefficients of the terms at equal distances on each side of it are the same; (iii) if n is odd there are two middle terms whose Newtonian coefficients are the same.

7. Calculate to the nearest shilling how much £1000 will amount to at compound interest (i) in ten years at 4 per cent per annum paid annually; (ii) in twenty years at 3 per cent per annum paid annually; (iii) in thirty years at 4 per cent per annum when the interest is added to the principal at half-yearly intervals.

8. Find to the nearest pound the compound interest on £100 for 100 years at 2 per cent per annum when the interest is added to the principal every quarter.

9. Write down the first five terms of the expansion of (i) $(1 - 2x)^{10}$; (ii) $(2 + x)^8$; (iii) $(3a - 4x)^6$; (iv) $(1 - \sqrt{x})^{20}$; (v) $(2x + 3)^7$.

10. Write down (i) the fourth term in the expansion of $(2 - 3x)^{10}$; (ii) the middle term in the expansion of $(a - bx^2)^8$; the term independent of x in $\left(2x - \dfrac{1}{2x}\right)^8$; (iv) the term independent of x in $(ax^2 + b/x)^6$; (v) the middle terms of $(2a - 3bx)^{10}$.

B.

Note.—Let V be the present value of a sum of £P for n years at compound interest i per pound, paid annually, then we know that

$$V = P(1 + i)^{-n}.$$

To calculate present values approximately we need, therefore, a rule for expanding $(1 + i)^{-n}$ when n is a positive whole number.

A suggestion towards such a rule will be found in Ex.

XXXVIII, Nos. 11-14, where it was shown that the approximate values of certain negative powers of the number $1 - i$ can be calculated by the formulæ

$$
\begin{aligned}
(1-i)^{-1} &= 1 + i + i^2 + i^3 + i^4 + \ldots \\
(1-i)^{-2} &= 1 + 2i + 3i^2 + 4i^3 + 5i^4 + \ldots \\
(1-i)^{-3} &= 1 + 3i + 6i^2 + 10i^3 + 15i^4 + \ldots \\
(1-i)^{-4} &= 1 + 4i + 10i^2 + 20i^3 + 35i^4 + \ldots
\end{aligned}
$$

The principle by which the coefficients are determined is derived from Ex. XXXVIII, No. 11. Let the coefficients in any one of the lines be, in order, $1, a_1, a_2, a_3, a_4, \ldots$ Then the corresponding coefficients in the next line are

$$1, (1 + a_1), (1 + a_1 + a_2), (1 + a_1 + a_2 + a_3),$$
$$(1 + a_1 + a_2 + a_3 + a_4), \ldots$$

If we call these coefficients $1, A_1, A_2, A_3, A_4, \ldots$ then it is obvious that

$$A_1 = 1 + a_1,\ A_2 = A_1 + a_2,\ A_3 = A_2 + a_3,$$
$$A_4 = A_3 + a_4, \text{ etc.}$$

Thus the coefficients in one expansion are derived from those of the preceding expansion by a law very similar to that of Stifel's table. Any coefficient is simply the sum of the coefficient directly before it in the same expansion *plus* the coefficient directly above it in the preceding expansion. For instance, 10, the coefficient of i^3 in the expansion of $(1 - i)^{-3}$ is the sum of the 6 immediately before it and the 4 immediately above it.

11. Starting with the row

$$1, 1, 1, 1, 1, 1, 1,$$

which may be regarded as the coefficients of the first seven terms of the expansion of $(1 - i)^{-1}$, make a table of the coefficients of the first seven terms in the expansion of $(1 - i)^{-n}$ where $n = 2, 3, \ldots 7$.

12. Now note that the expansion of $(1 + i)^{-n}$ can be obtained from the expansion of $(1 - i)^{-n}$ by substituting $-i$ for i. Use this principle to obtain the first seven terms in the expansions of (i) $(1 + i)^{-4}$, (ii) $(1 + i)^{-7}$.

13. Verify that the coefficients in any of the rows of the table in No. 11 obey the following law of derivation :—

$$1,\ 1 \times n,\ n \times \frac{n+1}{2},\ \frac{n(n+1)}{2} \times \frac{n+2}{3},\ \frac{n(n+1)(n+2)}{3\,!} \times \frac{n+3}{4}, \text{ etc.}$$

In other words, verify that if $n = 1, 2, 3, \ldots 7$,

$$(1 - i)^{-n} = 1 + ni + \frac{n(n+1)}{2\,!}i^2 + \frac{n(n+1)(n+2)}{3\,!}i^3 + \frac{n(n+1)(n+2)(n+3)}{4\,!}i^4 + \ldots$$

14. Assuming that the foregoing expansion holds good for a given value of n show by applying the rules

$A_1 = 1 + a_1$, $A_2 = A_1 + a_2$, $A_3 = A_2 + a_3$, etc.,

that it holds good also for $n + 1$.

15. Obtain an approximation-expansion for $(1 + i)^{-n}$ by substituting $-i$ for i in the expansion of No. 13.

Note.—There is an extremely important difference between the expansions of Nos. 4 and 15. The expansion of $(1 \pm i)^n$ has a definite number of terms, namely $n + 1$. It can, therefore, be used, not only for approximations, but also for calculating the *exact* value of $(1 \pm i)^n$ whatever be the value of i. On the other hand, the expansion of $(1 \pm i)^{-n}$ has no definite number of terms but is endless. Thus **it can be used for approximations only**. Moreover it can be used for this purpose only on condition that a definite number of terms is sufficient to give the result to the required degree of accuracy and that all the subsequent terms may be ignored.

The best way to make sure that this condition is fulfilled is to calculate the value of the complementary fraction (see p. 197) after a given number of terms. Thus we know (p. 199) in the case of the expansion of $(1 - x)^{-1}$ that after the term x^r has been calculated there is a complementary fraction $x^{r+1}/(1 - x)$. Since the value of this complement can easily be calculated, we can tell in any given case how much the value of the approximation is affected by ignoring it. Again we found (p. 215) that in the expansion of $(1 - x)^{-2}$ this complementary fraction is

$$\frac{(r+1)\,x^{r+1}}{1 - x} + \frac{x^{r+1}}{(1 - x)^2}$$

which can again be calculated, but not so easily. Similarly, the complementary fraction could be found for other values of n. It is to be observed, however, that this method of finding the degree of accuracy of an approximation-expansion is not only troublesome in practice but also theoretically unsound. For, as will be seen from the two examples just considered, the complementary fraction contains the very expression

which it is required to expand. It is better, therefore, to make use of a simpler, though indirect method.

16. Suppose in a given case that the expansion of $(1 - i)^{-n}$ has been calculated by the formula of No. 15 as far as the term containing i^{r-1}. Show that the next term is derived from this one by multiplying by the factor

$$\left(1 + \frac{n-1}{r}\right)i$$

and that this factor decreases as r increases.

17. The expression $(1 - 0{\cdot}05)^{-4}$ has been expanded as far as the fifth term (i.e. the term containing the fourth power of 0·05). What is the factor for calculating the sixth term?

18. Show that the sum of all the terms after the fifth is less than the sum of the series

$$0{\cdot}0000175 \times \{1 + 0{\cdot}08 + (0{\cdot}08)^2 + (0{\cdot}08)^3 + \ldots\}$$

i.e. less than 0·0000191.

19. Show that the error involved in using the first five terms of the expansion of $(1 + 0{\cdot}05)^{-4}$ as an approximation-formula is less than 0·0000163.

20. What sum of money invested at 5 per cent per annum compound interest, paid annually, will amount in four years to £1000? Use enough terms of the expansion to give the answer correctly to the nearest sixpence.

21. Find the upper limit of the error involved in neglecting all terms after the third in the expansion of (i) $(1 - 0{\cdot}03)^{-5}$, (ii) $(1 + 0{\cdot}03)^{-5}$.

22. I want to know approximately the present value of £100 due in five years at 3 per cent compound interest, paid annually. Calculate it by using three terms of the expansion of $(1 + 0{\cdot}03)^{-5}$ and determine within what amount the answer is correct.

23. Calculate the height of the ordinates where $x = -6$ in each of the curves of fig. 50 (p. 273). [It is, of course, useless to carry the approximation beyond the point at which the terms cease to yield results measurable in the graph.]

24. Write down

(i) the 4th term in the expansion of $(1 - 2x)^{-7}$;
(ii) the 8th term in the expansion of $(a + bx)^{-5}$;
(iii) the 5th term in the expansion of $(10 - 3x)^{-3}$.

C.

Note.—The preceding discussion has shown that if n is a whole number we may always make use of the approximation-formulæ

$$(1+i)^n = 1+ni+\frac{n(n-1)}{2!}i^2+\frac{n(n-1)(n-2)}{3!}i^3+ \ . \ . \ . \quad \text{(I)}$$

$$(1+i)^{-n} = 1-ni+\frac{n(n+1)}{2!}i^2-\frac{n(n+1)(n+2)}{3!}i^3+ \ . \ . \ . \quad \text{(II)}$$

It can easily be shown that these two formulæ may be regarded as special cases of a single formula

$$(1+i)^m = 1+mi+\frac{m(m-1)}{2!}i^2+\frac{m(m-1)(m-2)}{3!}i^3+ \ . \ . \ . \quad \text{(III)}$$

in which m may be either a positive integer (n) or a negative integer ($-n$).

25. Prove this statement by substituting $-n$ for m in formula III.

26. The formula $y = r^x$ which describes the "exponential curves" of fig. 50 (p. 273) can also be written

$$y = (1+i)^x$$

where $r = 1 + i$. If r is less than 2, i is less than 1. It is possible, therefore, to calculate y approximately by means of formula III for all integral values of x, positive and negative. But if the expansion gives the ordinates of the curves when $x = +1, +2, +3 \ . \ . \ . -1, -2, -3 \ . \ . \ .$ it seems probable that it will also give them when x has fractional values such as $+1{\cdot}3$, $+4{\cdot}2$, $-0{\cdot}6$, etc. Let each student select in one of the three curves of fig. 50 any two ordinates whose abscissæ are fractional numbers. Substitute the chosen values of i and x in the formula

$$(1+i) = 1+xi+\frac{x(x-1)}{2!}i^2+\frac{x(x-1)(x-2)}{3!}i^3+ \ . \ . \ .$$

carrying the calculation out until the terms cease to be of measurable magnitude. Compare the values thus obtained with the measured ordinates of the curves. They will be found to agree perfectly.

Note.—The investigation of No. 26 cannot, of course, be called a proof. A strict proof is, in fact, too difficult to be considered at this stage.[1] Nevertheless, the agreement between the lengths of measured ordinates selected at random

[1] It is given towards the end of Part II of this work.

and the values calculated by the formula leaves no reasonable doubt that the latter can be considered as an expansion of $(1 + i)^m$ for all values of m integral or fractional, positive or negative. In this statement it is to be understood that the value of i is such that the expansion is an approximation-formula in which a given degree of accuracy can be obtained by taking into account only a limited number of terms.

The statement that formula III can be used in this way for all values of m is called the **Binomial Theorem**.

27. Verify the Binomial Theorem by using it to calculate (i) $\sqrt{1{\cdot}1}$, (ii) $\sqrt[3]{1{\cdot}2}$, each to three places of decimals and comparing the results with those obtained by means of logarithms. [Note that $\sqrt{1{\cdot}1} = (1 + 0{\cdot}1)^{\frac{1}{2}}$, $\sqrt[3]{1{\cdot}2} = (1 + 0{\cdot}2)^{\frac{1}{3}}$.]

28. Find the sum of the first four terms in the expansion of

(i) $(1{\cdot}04)^{2{\cdot}5}$; (ii) $(0{\cdot}98)^{-6{\cdot}7}$;
(iii) $(1 + x)^{3{\cdot}2}$; (iv) $(1 - x)^{-4{\cdot}8}$.

29. Obtain by the Binomial Theorem a solution correct to 0·1 oz. of Ex. LII, No. 3.

30. Obtain by the Binomial Theorems answers to Ex. LII, No. 7, correct to the nearest fathom.

31. The pull in pounds weight needed to tow a certain canal boat at a speed of v miles per hour was found to follow the law

$$P = 32v^{1{\cdot}8}.$$

Calculate to the nearest pound the pull required when the speed is (i) 1·3 miles per hour; (ii) 0·8 miles per hour. Check your results by means of the curves of p. 273.

32. When air is suddenly compressed (for example in using a bicycle-pump) its temperature rises. The rule for the rise of temperature is

$$T + 273^\circ = (T_0 + 273^\circ)\left(\frac{p}{p_0}\right)^{0{\cdot}29}$$

where T_0 and p_0 are the original temperature and pressure of the air and T and p the temperature and pressure immediately after the compression, the temperature being measured in degrees centigrade. On a day when the temperature is 7° C. the air in a bicycle-tyre is compressed ten times in rapid succession, the pressure being increased by one-fifth at each stroke. Find to the nearest degree the temperature of the air in the tyre, assuming that there is no time for cooling to take place.

D.

33. In Ex. LIX it was shown that as n increases the value of $(1 + 1/n)^n$ approaches a certain number, "e," which it never quite reaches. It was shown by a graphic method that $e = 2{\cdot}71. \ . \ . \ .$ By writing out the first few terms of the expansions of the following expressions you should be able to find a very simple expansion which can obviously be regarded as an approximation-formula for e. Carry out the subsequent lines of your work upon the model shown in the first line.

$$(1 + 0{\cdot}1)^{10} = 1 + 10 \,.\, \frac{1}{10} + \frac{10\,.\,9}{2\,!}\cdot\left(\frac{1}{10}\right)^2 + \frac{10\,.\,9\,.\,8}{3\,!}\cdot\left(\frac{1}{10}\right)^3 + \frac{10\,.\,9\,.\,8\,.\,7}{4\,!}\cdot\left(\frac{1}{10}\right)^4 + \ . \ . \ .$$

$$= 1 + 1 + \frac{0{\cdot}9}{2\,!} + \frac{0{\cdot}72}{3\,!} + \frac{0{\cdot}504}{4\,!} + \ . \ . \ .$$

$(1 + 0{\cdot}01)^{100} =$
$(1 + 0{\cdot}001)^{1000} =$
$(1 + 0{\cdot}0001)^{10000} =$
$(1 + 0{\cdot}00001)^{100000} =$

34. Calculate e to three decimal places by means of the expansion

$$e = 1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \ . \ . \ .$$

35. In Ex. LIX it was proved that as n increases

$$\left(1 + \frac{a}{n}\right)^n \text{ approaches } e^a.$$

Find by the method indicated in No. 34 an approximation-formula for calculating the value of e^a.

36. Use the formula of No. 35 to calculate approximate answers to Ex. LIX, No. 9.

EXERCISE LXIX.

THE GENERALIZATION OF WALLIS'S LAW.

A.

1. Prove by the Binomial Theorem that if $y = ax^n$ then

$$\frac{\delta y}{\delta x} = nax^{n-1}$$

n being any positive whole number.

2. Prove similarly that if $y = ax^{-n}$ then

$$\frac{\delta y}{\delta x} = -nax^{-(n+1)}.$$

3. Show that the two preceding results are included in the statement that, if $y = ax^m$ then $\delta y/\delta x = max^{m-1}$, m being any whole number, positive or negative.

4. Write down the value of $\delta y/\delta x$ in the following cases :—

(i) $\mathbf{y = x^5}$. (ii) $\mathbf{y = 1/x^5}$. (iii) $\mathbf{y = 4x^7}$.
(iv) $\mathbf{y = 5/x^8}$. (v) $\mathbf{y = 5x^4 - 12x^3 + 2x^2 - 10x + 1}$.
(vi) $\mathbf{y = x^6 - 2x^4 + 3x^2 - 4 + 3/x^2 - 2/x^4 + 1/x^6}$.

5. Write down the value of $\delta^2 y/\delta x^2$ in the case of No. 4 (v) and (vi).

6. Write down the value of $\delta^3 y/\delta x^3$ in the use of No. 4 (i) to (iv).

Note.—The foregoing results are extremely important because they show that Wallis's Law holds good for all integral values of m in $y = ax^m$. This conclusion follows from the fact that the binomial expansion has been *proved* in the preceding exercise to hold good as an approximation-formula whenever m is integral. We also saw good reason to think that it holds good when m is fractional. If this were proved to be the case Wallis's Law would be proved to be true for all values of m. Fortunately, in order to derive a differential formula from $y = ax^m$ it is sufficient to know the first

term and the coefficient of x. We need not know the coefficients of the higher powers of x if, as usual, these higher powers may be neglected in forming the differential formula. Now it is easy to prove (*with a certain assumption*) that the first two terms of the expansion of $(1 + x)^{p/q}$ are

$$1 + \frac{p}{q}x.$$

That is, that they follow the law of the Binomial Theorem. The proof is as follows :—

Let $\qquad (1 + x)^{p/q} = 1 + x'$

then $\qquad (1 + x)^p = (1 + x')^q$.

Since p and q are both integers, positive or negative, it follows that

$$1 + px + \text{R} = 1 + qx' + \text{R}'$$

when R and R′ symbolize the remaining terms of the two expansions. *Assuming*, then, that (as on p. 348) when x is small enough these remainders are negligible, we have

$$px = qx'$$

or $$x' = \frac{p}{q}x$$

whence $$(1 + x)^{p/q} = 1 + \frac{p}{q}x + \ . \ . \ .$$

7. Prove that the formula $\delta y/\delta x = max^{m-1}$ holds good when $m = p/q$, p and q being integers of either sign.

8. Write down the value of $\delta y/\delta x$ when

(i) $y = x^{\frac{1}{2}}$; (ii) $y = \sqrt[3]{x^2}$; (iii) $y = 1/\sqrt[5]{x^3}$;
(iv) $y = 10x^{2\cdot4}$; (v) $y = 4x^{-1\cdot8}$.

9. Show that

$$(a + x + h)^m = (a + x)^m + mh \,.\, (a + x)^{m-1} + \text{R}$$

where R is a series of terms in each of which h occurs as a factor in a power above the first. Hence prove that if $y = (a + x)^m$, $\delta y/\delta x = m(a + x)^{m-1}$. Write down the corresponding result when $y = (a - x)^m$.

10. Write down a formula for $\delta y/\delta x$ when

(i) $y = (1 + x)^3$; (ii) $y = 1/(2 - x)^4$;
(iii) $y = 4(x - 2\cdot5)^{1\cdot8}$; (iv) $y = (2x + 3)^3$;
(v) $y = (ax + b)^m$; (vi) $y = 1/\sqrt[3]{(3x - 4)}$.

B.

Note.—In fig. 61 the whole space under the curve is supposed to be divided into strips of equal width. Some of the

strips are shown as specimens. Let the abscissa of any point P[1] be x and let the area under the curve, from the y-axis up to P, be A. Then, just as NN′ is called an increment of x, so the area of the strip PNN′P′ may be called the corresponding increment of A. This area will, in general, be different from that of the rectangle PNN′Q′, but it is evident that by making the strips sufficiently narrow the difference can be reduced to less than c per cent of the area of the rectangles in every case, however small c may be. Let w be the greatest width for which the difference is less than c per

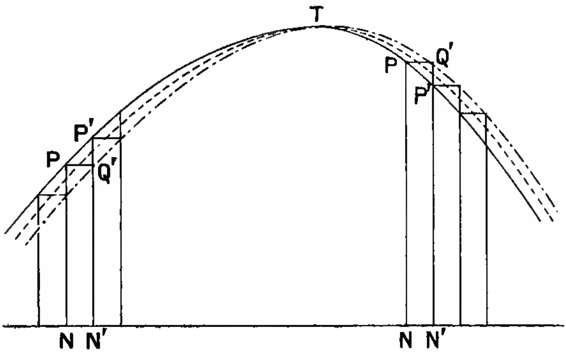


FIG. 61.

cent. Then for still smaller widths the increment of the area is, proportional, within less than c per cent, to the increment of x. That is, in order to calculate the area of the strip to the specified degree of accuracy we need not take account of any powers of the width except the first.

In these circumstances we may, in accordance with previous definitions, call NN′ the differential of x, and the area of the strip the differential of A. Moreover (since PN = y), we have the differential formula

$$\delta A = y\delta x \text{ or } \delta A/\delta x = y.$$

Now if the ordinate-function of the curve is known we can substitute it for y in this formula and find the primitive.[2]

[1] Two points on the curve are marked P. The argument applies indifferently to either of them.

[2] For example, if $y = 4x^3$ we have $\delta A/\delta x = 4x^3$, whence $A = x^4 + p$ where p is the usual undetermined constant.

Assuming that we have additional information sufficient to determine the constants, we have now a formula expressing A as a function of x. As was shown in the discussion preceding Ex. XLVII, values of A obtained from this formula cannot be more than $2c$ per cent in error. Moreover, since the ordinates may be supposed as close together as we please, $2c$ may be considered smaller than any number that can be named. In other words, the primitive deduced from $\delta A = y\delta x$ may be regarded as the area-function of the curve.

Fig. 61 illustrates this conclusion. Across the "column-graph" composed of the rectangles a broken curve is drawn in such a way that the area under it between any two bounding ordinates of the rectangles is exactly the same as the total area of the rectangles between those same ordinates. Let A′ be the area-function of this curve. Then the successive rectangles are the *exact* first differences of the successive values of A′ at intervals of δx. Now by hypothesis the curve whose area-function is the primitive of $\delta A/\delta x = y$ may, in any part of the figure, include anything up to c per cent more or c per cent less than the dotted curve includes. That is, the curve calculated from the differential formula *may* run like the firm curve or like the curve marked out by alternate dots and dashes in fig. 61; but in no region of the figure will the space between its extreme possible positions be more than $2c$ per cent of the area of the corresponding rectangles. But even in the most unfavourable case by making δx small enough $2c$ may be made as small as we please. That is, the added curves may be made to close up to the given curve so as to become indistinguishable from it. This is, of course, simply another way of saying that the primitive deduced from the formula $\delta A/\delta x = y$ may be regarded as the area-function of the curve.

11. Find the area-function when the ordinate-function is (i) $y = 3x^4$, (ii) $y = x^{\frac{2}{3}}$, (iii) $y = 3x^{2 \cdot 4}$. [Find the primitive of $\delta A = y\delta x$ by Wallis's Law, and determine the constant by the consideration that when x is zero A is zero.]

12. Find the area between the x-axis, the curve

$$y = 3x^4 - 2x^3 + x - 7$$

and the ordinates where $x = -2$ and $x = +10$.

13. Find the area-functions of the curves whose ordinate-functions are

(i) $y = (2 + x)^{\frac{1}{2}}$; (ii) $y = \sqrt[3]{(2x - 1)^2}$;
(iii) $y = 1/(1 - 3x)^4$; (iv) $y = 1/(3x + 1)^{3.4}$.

14. "Wallis's Law gives the primitive of $\delta y/\delta x = x^{m-1}$ for all values of m except $m = 0$." Why does the law break down for this value?

C.

Note.—Imagine a point to be moving with variable speed along a straight line. At time t let its distance from a fixed origin, O, be s, t being measured from the moment when it passes through O. Suppose it to travel in time h from a certain point P to another point P′. Then we have

average velocity between P and P′ = PP′/h.

Since the point is moving with variable speed the value of PP′/h will depend upon the length of PP′. Let P be fixed but let P′ be taken successively nearer and nearer to it. Then the average velocity will be different for each of these different positions of P′. It is obvious, however, that when PP′, and therefore h, are small enough the subsequent values of the average values will not differ by more than c per cent where c is any number as small as we please. That is, equal distances within the range PP′ will be covered in times which are equal, to the specified degree of exactness, however high that degree may be. If v be this final value of the average velocity, we can write δt for h and δs for PP′ and we have the differential formula

$$\delta s = v\delta t.$$

In fig. 61 let t be measured along the base and let the ordinates at equal distances δt have for their heights the corresponding values of v. Then the areas of the successive rectangles, such as PNN′Q′, will be the successive values of $v\delta t$ or δs. It follows that the total area of the column-graph up to the ordinate t will differ by less than c per cent of its value from the area which measures the actual distance covered by the moving point in time t. That is to say, this distance will be measured by the area under the dotted curve in the figure. But (as in the Note before No. 11) if δt is taken small enough this area will also differ by less than c per cent from that under the curve corresponding to the primitive of $\delta s/\delta t = v$, when for v we substitute the function by which it can be calculated from t. In the most unfavour-

able case the curve giving the true values of s and that which corresponds to the primitive of $\delta s/\delta t = v$ may lie on opposite sides of the dotted curve, but they will include between them an area which is less than $2c$ per cent of the area of the column-graph.[1] Thus the primitive of $\delta s/\delta t = v$ enables us to calculate the actual distance covered by the moving point to a degree of accuracy limited only by the degree of exactness adopted in measuring v.

The proper way to describe v is to call it **the average velocity of the point during a short time δt after it passes a point P**. This definition warns us that it is not an absolutely fixed value but only one whose variation is limited within a range of given minuteness. For brevity it may be called **the velocity of the moving point at P**, but the full meaning of this expression should be borne in mind.

15. A point moves along a straight line in such a way that its distance s from a fixed point O is given by the relation

$$\text{(i) } s = 2/(t+1)\,; \qquad \text{(ii) } s = +\,3{\cdot}4 + 5t^{1\cdot 8}\,;$$
$$\text{(iii) } s = -\,10t + 4t^{2\cdot 5}\,; \qquad \text{(iv) } s = -\,10/(2t+3)^{\frac{3}{4}}.$$

Write down a formula giving the velocity ($v = \delta s/\delta t$) at time t in each case. Find the velocity of the point in each case when $t = 0$ and when $t = 1$.

16. The velocity of a point moving in a straight line is given at different times by one of the following formulæ. In each case its position is measured by its distance (s) from the point which it occupies when $t = 0$. Find the position-formulæ :—

$$\text{(i) } \delta s/\delta t = +\,2 - 4t^{\frac{2}{3}}\,; \qquad \text{(ii) } \delta s/\delta t = 1 + 2t + 3t^2\,;$$
$$\text{(iii) } v = 100/\sqrt{(1+5t)}\,; \qquad \text{(iv) } v = -\,20/\sqrt[3]{(2-3t)}.$$

Calculate in each case the position of the point after it has been moving for 1 unit of time.

Note.—If a point is moving along a line the number $\delta^2 s/\delta t^2$ measures what should, strictly speaking, be called the **average acceleration of its velocity during a short time δt** after it passes any given point situated s from the origin. Less correctly this number is sometimes called the **acceleration at P**.

[1] It should be noted that neither of these curves need pass through the corners of the rectangles as in fig. 61.

17. Write down formula expressing the acceleration of the point in No. 15, (i) and (ii) and in No. 16, (i) and (ii).

18. The acceleration of a point is given by the formulæ

(i) $\delta^2 s/\delta t^2 = +10 - \frac{1}{2}t$; (ii) $\delta^2 s/\delta t^2 = -5 + \sqrt{t}$;
(iii) $a = 3(1 + 4t)^{1 \cdot 5}$; (iv) $a = -4/(1 + 2t)^3$.

The time is in each case measured from the moment when the velocity is zero. Write down the velocity-formulæ.

19. Write down also the position-formulæ corresponding to the acceleration-formulæ of No. 18, given that the position of the moving point is measured from the point which it occupies when $t = 0$.

20. A point moves along a line with constant acceleration p. Show that its distance from any fixed point in the line is a parabolic function of the time.

D.

21. Write down the formula which gives the gradient at any point of the curves specified in No. 11.

22. Find, in degrees, the inclination to the x-axis of the tangent at the point on the curve where $x = +1$ in each of the cases of No. 11.

23. Calculate the ordinates at these points. Knowing the slope of each tangent and the co-ordinates of its point of contact with its curve, write down the formula which describes it. [See Ex. XXXIX, No. 16.]

24. Find where each of the tangents of No. 23 cuts the x-axis. Call this point T, the point of contact P, and the foot of the ordinate through P N. Then TN is called the **subtangent.** Calculate the length of the subtangent in each of the cases of No. 23.

25. Let p be the abscissa of any point P of the parabola $y = a^2x^2$. What is the ordinate of P? Show that the formula of the tangent through P is

$$y = a^2p(2x - p).$$

Hence prove that the subtangent is one half of the abscissa. Illustrate by a diagram.

26. The parabola of No. 25 is turned into the position in which its formula is

$$y = \frac{1}{a}\sqrt{x}.$$

Show that the subtangent at any point is bisected by the vertex.

27. Prove that the lines $y = mx$ and $y = -\frac{1}{m}x$ are at right angles for all values of m. Show that it follows that the lines $y - b = m(x - a)$ and $y - b' = -\frac{1}{m}(x - a')$ are also at right angles, whatever be the values of a, a', b, b'.

28. The straight line drawn at right angles to the tangent to a curve at its point of contact is called a **normal** of the curve. Write down the formulæ of the normals in the case of each of the tangents of No. 23.

29. Let the normal at a point P meet the x-axis in the point Q. Then NQ is called the **subnormal**. Calculate the length of the subnormal in each case of No. 28.

30. Show that in the case of a parabola in the position corresponding to $y = a^2x^2$ the subnormal is proportional to the cube of the abscissa. Show that in the position corresponding to $y = \frac{1}{a}\sqrt{x}$ it is constant.

ANSWERS TO THE EXAMPLES.

*** 1. The solidus notation for division is used throughout these answers for convenience in printing. The pupil should be taught both to read and to use it freely, but it should be remembered that the fractional notation is often more appropriate.

2. The answers to trigonometrical problems in Sections I. and II. are those which are obtained by using the three-figure tables on pp. 107 and 111.

3. The answers to graphic problems are in some cases given to a higher degree of approximation than the average pupil is likely to reach. In other cases slightly different answers might be obtained by equally competent draughtsmen.

4. The references above each set of answers are to the chapters in *The Teaching of Algebra.*

SECTION I.

EXERCISE I.

See ch. III. ; ch. VI., § 3.

1. $A = lb$. 2. $b = A/l$.
3. (i) $C = np$; (ii) $p = C/n$.
4. $C = Nc/n$. 5. $p = 12s$. 6. $s = p/12$.
7. (i) $s = 20L$; (ii) $p = 240L$.
8. (i) $L = s/20$; (ii) $L = p/240$.
9. (i) $A = 3lb$. ; (ii) $A = nlb$.
10. (i) $A = s^2$; (ii) $A = ns^2$.
11. (i) $V = Ah$; (ii) $V = lbh$.
12. (i) $d = V/A$; (ii) $d = V/lb$.
13. $C = nc/N$. 14. $C = nc/20N$. 15. $t = p + 1$.
16. (i) $P = 12s. + p$; (ii) $S = 20L + s$; (iii) $P = 240L + p$.
17. $C = 6 + \frac{1}{2}n$. 18. $A = 2\frac{1}{2} + \frac{1}{6}n$. 19. $t = \frac{1}{4}w + \frac{1}{3}$.
20. $a = A - 27$. 21. $a = \frac{1}{2}A - 3$. 22. $p = 30 - \frac{3}{4}n$.
23. $G = np - NP$. $L = NP - np$.
24. $I = Pnr/100$. 25. $A = P + Pnr/100$.

EXERCISE II.

See ch. IV., §§ 2-5.

A.

2. (i) CD; (ii) equal; (iii) AB.
8. 11.52 a.m. (i) 7·52 cms. ; (ii) 10·46 cms.

B.

12. (i) 0·75 inch ; (ii) 1·65 inches ; (iii) $3\frac{1}{3}$ lb. ; (iv) $13\frac{1}{3}$ lb.
13. (i) 18 cms. ; (ii) 148 cms. ; (iii) 2·5 seconds ; (iv) 3·54 seconds.
14. (i) 35·2 cms. ; (ii) 65·6 cms.
15. (i) 15s. ; (ii) 21s. 6d. ; (iii) 39s.
16. (i) 50° ; (ii) 12.2 p.m. ; (iii) 5 a.m., 7.6 p.m.
17. (i) 10·83 gms. ; (ii) 25·86 cms.
18. (i) 3 feet $3\frac{1}{2}$ inches ; (ii) 0·97 second.
19. (i) 6.4 p.m. ; (ii) 8.22 p.m. ; (iii) April 19.
20. (i) £43 13s. ; (ii) £55 15s.

C.

22. (i) 3·35 acres; (ii) 16·1 acres.
25. (i) 8680; (ii) 7480; (iii) 1340; (iv) 9th week; (v) 12th week.

D.

28. See ch. IV., § 5.

29. See ch. IV., § 5.

EXERCISE III.

See ch. III.; ch. VI., §§ 1, 2.

A.

1. (i) $W = b + 37m$; (ii) $W = b + nm$; (iii) $W = b + Vi$; (iv) $l = 8t_1 + 5t_2$; (v) $l = \frac{3}{8}n_1 + \frac{7}{8}n_2$; (vi) $l = n_1t_1 + n_2t_2$; (vii) $W = n_1p + n_2s + n_3h$; (viii) $P = n_1p + 12n_2s + 30n_3h$.
2. (i) $S = 50 + 5t$; (ii) $S = S_0 + it$.
3. (i) $s = 14 + 3t$; (ii) $s = s_0 + it$.
4. (i) $S = 35 - 2t$; (ii) $S = S_0 - rt$.
5. (i) $V = 45 - 10t$; (ii) $V = V_0 - rt$; (iii) $V = V_0 - r_1t_1 + r_2t_2$.
6. (i) $R = f + n_1m - n_2r$; (ii) $R = n_2r - f - n_1m$.
7. (i) $A = HB - hb$; (ii) $A = HB - 3hb$; (iii) $A = HB - nhb$.
8. (i) $V = 40 + 1{\cdot}7t$; (ii) $V = 40 + 3{\cdot}4t$; (iii) $V = 40 - 2{\cdot}1t$; (iv) $V = V_0 + st$; $V = V_0 + 2st$; $V = V_0 - wt$.
9. In (i) 2 is added to each term to obtain the next term. To obtain the first term 2 must be added to 1. Hence
$$T_n = 1 + 2n.$$
The formulæ for T_n obtained in this way are :—
(ii) $3 + 5n$; (iii) $17\frac{1}{2} + 3\frac{1}{2}n$; (iv) $0{\cdot}7 + 0{\cdot}7n$; (v) $\frac{7}{24} + \frac{3}{8}n$; (vi) $100 - 6n$; (vii) $20 - 1{\cdot}8n$.
10. 21, 53, $52\frac{1}{2}$, 7·7, $4\frac{1}{24}$, 40, 2.
11. (i) $t = V/s$; (ii) $t = V/2s$; (iii) $t = V/(2s + w)$.
12. (i) $t = (V - 40)/s$; (ii) $t = (V - 40)/2s$; (iii) $t = (V - 40)/(2s + w)$.
13. (i) $A = lb$; (ii) $l = A/b$; (iii) $b = A/l$; (iv) $W = A/w$; (v) $A = W/w$; (vi) $w = W/lb$; (vii) $A = \frac{1}{2}bh$; (viii) $W = \frac{1}{2}wbh$; (ix) $W = 1{\cdot}7\ bh$; (x) $h = W/1{\cdot}7b$.
14. (i) $a = V/n$; (ii) $a = V/(n + 1)$; (iii) $a = V/(2n + 1)$; (iv) $a = V/(n - 2)$; (v) $a = V/(2n - 5)$.
15. (i) $s = (w_1 - w_2)/150$; (ii) $s = (w_1 - w_2)/n$.
16. $f = (l - 5w)/n$.
17. (i) $n = (N + 1)/2$; (ii) $n = N/2$; (iii) $n = (N_1 - N_2)/2 - 1$; (iv) Yes.
18. (i) $c = n/6$; (ii) $c = np/12$.
19. (i) $b = A/n$; (ii) $c = A/3n$.
20. (i) $b = A/n$; (ii) $b = A/22n$.

B.

21. (i) $i = V_1(v_2 - v_1)/v_1$;
(ii) $V_2 = V_1 v_2/v_1$ or $= V_1 + V_1(v_2 - v_1)/v_1$.
22. (i) $d = (r_2 - r_1)R_2/r_2$; (ii) $R_1 = R_2 r_1/r_2$.
23. (a) $S = l/n + t/n$; (b) $S = (l + t)/n$.
24. (a) $l = h/n + b/n$; (b) $l = (h + b)/n$.
25. (a) $S = p/n - d/n$; (b) $S = (p - d)/n$.
26. $S_1 = En_1/(n_1 + n_2)$; $S_2 = En_2/(n_1 + n_2)$.
27. (i) $i = (b - d)t$; (ii) $P = P_0 + (b - d)t$;
(iii) $P = P_0 - (d - b)t$.
28. (i) $d = (s_1 + s_2)t$; (ii) $d = d_0 + (s_1 + s_2)t$;
(iii) $d = (s_1 - s_2)t$; (iv) $d = d_0 + (s_1 - s_2)t$;
(v) $d = d_0 - (s_1 - s_2)t$.
29. (i) $r = R/(m_2 - m_1)$; (ii) $r = R/5280(m_2 - m_1)$;
(iii) $r = R/52{\cdot}8(m_2 - m_1)$.
30. (i) $r = R/(d_2 - d_1)$; (ii) $r = R/12(d_2 - d_1)$.
31. (i) $I = n(r - g)$; (ii) $T = \frac{7}{120}n(r - g)$;
(iii) $I = n(r - g)/c$; (iv) $T = \frac{7}{120}n(r - g)/c$.
32. (i) $i = (d_2 - d_1)T/t$; (ii) $d = d_1 + (d_2 - d_1)\,T/t$;
(iii) $d = d_1 - (d_1 - d_2)T/t$; (iv) $T = d_1 t/(d_1 - d_2)$.
33. (i) $n = 6w/5$; (ii) $n = 3w/2$.
34. (i) $n = 8l/7$; (ii) $n = 96l/7$;
(iii) $n = 4l/5$; (iv) $n = 48l/5$.
35. (i) $n = 3A/2l$; (ii) $n = 4A/3l$; (iii) $n = 6A/5l$.
36. (i) $C = 30 + 2n$; (ii) $C = 2\frac{1}{2} + \frac{1}{6}n$; (iii) $n = (C - 30)/2$;
(iv) $n = 6(C - 2\frac{1}{2})$.
37. (i) $b = 120\left(T - \dfrac{m}{80}\right)$; (ii) $m = 80\left(T - \dfrac{b}{120}\right)$;
(iii) $n = m + 120\left(T - \dfrac{m}{80}\right)$; (iv) $n = b + 80\left(T - \dfrac{b}{120}\right)$.
38. (i) $n_1 = 2(20T - n_2)$; (ii) $n = n_2 + 2(20T - n_2)$ or $n = 40T - n_2$;
(iii) $n_1 = (20T - n_2 s)/a$; $n = n_2 + (20T - n_2 s)/a$
or $n = \{20T - (s - a)n_2\}/a$.
39. (i) $P = W + nw$; (ii) $w = (P - W)/n$.
40. (i) $d = 24 - \frac{7}{2}t$; (ii) $T = \frac{2}{7}(24 - \frac{7}{2}t)$;
(iii) $d = d_0 - rt$; $T = (d_0 - rt)/r$.

EXERCISE IV.

See ch. VI., §§ 1, 3.

A.

1. (i) 1, 3, 5, 7, 19; (ii) 1, 5, 9, 13, 37; (iii) 94, 88, 82, 76, 40;
(iv) 4, 16, 28, 40, 112; (v) 3, $5\frac{2}{3}$, $8\frac{1}{3}$, 11, 27; (vi) 0, 1, 4, 9, 81;
(vii) 4, 15, 30, 49, 247; (viii) 6, 7, 10, 15, 87;
(ix) 4, 44, 156, 376, 5980; (x) $2\frac{1}{2}$, 2, $1\frac{3}{4}$, $1\frac{3}{5}$, $1\frac{3}{11}$;
(xi) 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{19}$; (xii) 1, $\frac{4}{15}$, $\frac{2}{15}$, $\frac{4}{49}$, $\frac{4}{247}$.
2. The third, £235. 3. (ii) (a) $7\frac{1}{4}$ inches; (b) $6\frac{5}{8}$ inches.

4. (i) 1·77 tons; (ii) 10·23 tons. 5. (i) 0·34 lb.; (ii) 34·3 lb.
6. (i) 6·1 miles; (ii) 19·52 miles; (iii) 24·4 miles.
7. (i) 1·6 feet; (ii) 1·35 feet.
8. (i) 1 ft./min.; (ii) 4 ft./min.; (iii) 100 ft./min.
10. (i) 7·71 miles; (ii) 9·13 miles; (iii) 10·57 miles; (iv) 17·25 miles; (v) 33·86 miles.
11. 61·28 feet. 12. (i) 20·04 ft./min.; (ii) 70·24 ft./min.
14. (i) 11·7 feet; (ii) 14·1 feet. 15. $245\frac{1}{3}$ feet; 115 feet.

B.

17. (i) 4·53 cwt.; (ii) 3·07 cwt.; (iii) 5 cwt.
18. (i) 822·8 gallons; (ii) 495·72 gallons.
19. (i) 0·512 ton; (ii) 7·2 tons; (iii) 7·2 tons; (iv) Yes.
21. (i) 1·812 tons; 2·813 tons; (ii) 3·16 inches; 3·94 inches.
22. (i) 10 tons; (ii) 37 tons.
23. (i) 3·2 inches; (ii) 2·32 inches; (iii) 27·04 inches.
24. 0·76 ton. 25. Oral. 26. (i) $\frac{1}{3}$ foot; (ii) 4 inches.
27. 259·2 lb. 28. 150·04 feet. 29. 0·97 foot.

C.

30. (i) 20,160 lb.; (ii) 27,337·5 lb.
31. (i) 4·5 lb.; (ii) 3·75 lb.; (iii) 2·85 lb.
32. (i) 749·8; (ii) 1075·2. 33. (i) 5040; (ii) 12960.

EXERCISE V.

See ch. VI., § 4; ch. VII., A.

1. Square feet: (i) 520; (ii) 1280; (iii) 990; (iv) 580.
2. Square feet: (i) 1500; (ii) 680; (iii) 396; (iv) 630; (v) 1040; (vi) 1825.
3. Square feet: (i) 1500; (ii) 2600; (iii) 1600; (iv) 1540; (v) 6000; (vi) 2200.
4. $A = (a + b + c)d$. 5. $A =$ (i) $(a + b)c$; (ii) $a(b + c)$; (iii) $(a + 2b)d$; (iv) $(3a + b)c$; (v) $a(2b + c)$; (vi) $a(b + 4c)$.
6. (i) 360; (ii) 40; (iii) 6s. 8d.
7. (i) $A = a(b - c)$; (ii) $C = pa(b - c)/9$.
8. (i) $A = a(b - c - d)$; (ii) $C = pa(b - c - d)/9$.
9. (i) $A = a(b - c)$; (ii) $C = pa(b - c)$; (iii) $C = pa(b - c)/9$.
10. 175 square feet. 11. $A = (a - b)c$. 12. $C = (a - b)cl/9$.
13. Square feet: (i) 2000; (ii) 3200; (iii) 4560; (iv) 4600; (v) 4950.
14. $A =$ (i) $(2a - b)c$; (ii) $a(2b - c)$; (iii) $(3a - b)c$; (iv) $(\frac{3}{2}a - c)b$; (v) $(a - \frac{3}{4}c)b$.
15. $A =$ (i) $(a + 3b)c$; (ii) $2(a + b)c$; (iii) $2(a + \frac{3}{2}b)c$ or $(2a + 3b)c$; (iv) $(a + b + c)d$; (v) $a(a + 2b)$; (vi) $a(a - 2b)$; (vii) $(a + c)b$; (viii) $2(a - b)c$; (ix) $(3a - 2b)b$.

16. (i) 86; (ii) 83; (iii) 39; (iv) 260; (v) 280; (vi) 56; (vii) 209; (viii) 1540; (ix) 6; (x) $\frac{1}{7}$.
17. Exp. = (i) $a^2(b + c)$; (ii) $(pr - q)q$; (iii) $(p - r)q^2$; (iv) $(al + bm)lm$; (v) $a^2(a - p)$; (vi) $(pa + q^2)a^2$; (vii) $(ab + bd - de)c$; (viii) $a(2b + 3c)$; (ix) $pq(\frac{3}{2}p - 4q)$; (x) $ab(ab - 2)$; (xi) $pq(ap - 1)$; (xii) $(ar^2 - 2br + 1)r$.
18. (i) $A = l(4l - 5d)$; (ii) $M = ap(\frac{1}{2}a + \frac{1}{3}q)$; (iii) $V = \frac{\pi}{3}r^2(4r + h)$; (iv) $P = abc(a - b + c)$; (v) $Q = ab(2a^2 + 3b^2 - 1)$; (vi) $T = (ap^2 - 2bpq + 3q^2)q$; (vii) $W = 2rw(\pi r - 2a)$; (viii) $B = 3mn(4m + 3n - 1)$; (ix) $C = a(2a^2 - 5a + 1)$; (x) $V = \pi ac(3a + 4b)$.
19. $W = aw(a + 2b)$. 20. $W = (3a - 2b)bw$.

EXERCISE VI.

See ch. VI., § 4; ch. VII., B.

1. Square feet: (1) 15,800; (ii) 5760; (iii) 18,400; (iv) 2232.
2. (i) 180 sq. cms.; (ii) 44 sq. inches; (iii) $52\frac{1}{2}$ sq. inches; (iv) 247·2 sq. inches.
3. 720 c.cs. 4. (i) $V = (a + b)(a - b)l$; (ii) $W = (a + b)(a - b)wl$.
5. $w = W/(a + b)(a - b)l$. 6. $A = (a + 2b)(a - 2b)$.
7. $A = (a + 3b)$. 8. $A = (a + 2b)(a - 2b)$.
9. (i) $W = (a + 3b)(a - 3b)w$; (ii) $W = (a - 2b)(a + 2b)wl$; (iii) $w = W/(a - b)(a + b)$.
10. $c = 20C/(a - b)(a + b)l$.
11. Expr. = (i) $(p + q)(p - q)$; (ii) $(2a + b)(2a - b)$; (iii) $(m + 3n)(m - 3n)$; (iv) $(6a + 5b)(6a - 5b)$; (v) $(pa + b)(pa - b)$; (vi) $(u + vt)(u - vt)$; (vii) $(pu + vt)(pu - vt)$; (viii) $(ab + 4c)(ab - 4c)$; (ix) $(a + 4)(a - 4)$; (x) $(9 + b)(9 - b)$; (xi) $(pq + 5)(pq - 5)$; (xii) $(1 + mn)(1 - mn)$.
12. (i) $p(a + b)(a - b)$; (ii) $a(2p + 3q)(2p - 3q)$; (iii) $\pi(r_1 + r_2)(r_1 - r_2)$; (iv) $a(a + b)(a - b)$; (v) $(p + t)(p - t)t$; (vi) $(2p + 5t)(2p - 5t)t$; (vii) $3(2a + 3b)(2a - 3b)$; (viii) $(a + 6b)(a - 6b)b$; (ix) $2(2a + 5)(2a - 5)$; (x) $p(2p + 3)(2p - 3)$; (xi) $(ab + c)(ab - c)$; (xii) $a(3ab + 1)(3ab - 1)$.
13. (i) $(a + b)(a - b + 3)$; (ii) $(a - b)(a + b + 7)$; (iii) $(a + b)(a - b - 4)$; (iv) $(a - b)(a + b - p)$; (v) $(a + b)(a - b + 1)$; (vi) $(a - b)(a + b + 1)$; (vii) $2(a - b)(a + b + 3)$; (viii) $(a + b)(2a - 2b - 3)$; (ix) $(p - q)(ap + aq + b)$; (x) $(p + q)(ap - aq - b)$.
14. (i) $(a + b + c)(a + b - c)$; (ii) $(p - q + r)(p - q - r)$; (iii) $(a + b + 2)(a + b - 2)$; (iv) $(p - q + 3)(p - q - 3)$; (v) $(p + q + 3r)(p + q - 3r)$; (vi) $(a - b + 4c)(a - b - 4c)$; (vii) $(2u + 2v + w)(2u + 2v - w)$;

(viii) $(3u - 3v + 2w)(3u - 3v - 2w)$; (ix) $(2a + b)a$;
(x) $a(a + 2b)$; (xi) $a(a - 2b)$; (xii) $(2b - a)b$;
(xiii) $(2a + 3b)(2a + b)$; (xiv) $(6a - b)(4a - b)$;
(xv) $(a + 3b)(a - b)$; (xvi) $(a + 2b)(a - 4b)$;
(xvii) $4(5p + 2q)(2q - p)$; (xviii) $(5p - 2q)(5p - 8q)$.
15. (i) $(2a - b)b$; (ii) $(2p - q)q$; (iii) $3(2r - 3)$;
(iv) $6{\cdot}7(2p - 6{\cdot}7)$; (v) $aq(2p - q)$; (vi) $\pi w(2r - w)$;
(vii) $3b(2a - 3b)$; (viii) $rt(2s - rt)$; (ix) $pb(2a - pb)$;
(x) $aq(2p - aq)$.
16. $A = (2a + b)b$. 17. $A = a(a + 2b)$.
18. $A = 4a(a + 3b)$. 19. $A = 4p(p + 3q)$.
20. $A = (p + 11q)(p - 7q)$. 21. $A = 4b(3a + b)$.
22. $A = (2a - b)b$. 23. 220 sq. cms.
24. (i) $A = \pi(r_1 + r_2)(r_1 - r_2)$; (ii) $W = \pi w(r_1 + r_2)$;
(iii) $V = \pi l(r_1 + r_2)(r_1 - r_2)$; (iv) $W = \pi lw(r_1 + r_2)(r_1 - r_2)$;
(v) $w = W/\pi l(r_1 + r_2)(r_1 - r_2)$.
25. $A = \pi(a + 2b)(a - 2b)$.

EXERCISE VII.

See ch. VI., § 5; ch. VIII., A.

1. (i) 4·8 yards; (ii) 7·9 inches; (iii) 9·7 cms.; (iv) 8·485 feet;
(v) 1·81 miles; (vi) 5·36 feet; (vii) 9·01 yards;
(viii) 1·01 miles.
2. (i) 18·7; (ii) 39·5; (iii) 20·4; (iv) 0·207; (v) 0·0171.
3. 348·4 miles. 4. 241·5 miles. 7. 28·1 miles. 8. 89 miles.
9. 143·6 miles. 10. 10·9 ft./min. 11. $18\frac{3}{4}$ seconds.
12. $R = \sqrt{(a/\pi)}$. 13. (i) 6·34 inches. (ii) 0·761 foot.
14. 196·3 miles; 136·3 miles. 15. $R = \sqrt{(V/\pi L)}$.
16. 0·42 cm. 17. 0·076 cm.

EXERCISE VIII.

See ch. VI., § 7; ch. VIII., B.

1. 2·82, 3·46, 2·44, 6·23, 9·96, 23·85, 1·66, 0·72, 0·71, 0·64, 11·09.
2. 0·705, 2·88, 2·135, 0·69, 0·479, 0·851, 0·403, 7·02.
3. $2\sqrt{2}$, $\frac{1}{5}\sqrt{5}$, $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}\sqrt{15}$, $\sqrt{26}/13$, $\frac{a}{2}\sqrt{2}$, $\frac{1}{4}\sqrt{5p}$, $\frac{a}{3}\sqrt{3}$.
4. (i) 8; (ii) 1·28; (iii) 17·23; (iv) 2·02; (v) 0·055; (vi) 11·2;
(vii) 0·576; (viii) 3·875; (ix) 5·47; (x) 6·31.
5. $a\sqrt{2}$. 6. $a\sqrt{3}$. 7. $a\sqrt{10}$. 8. $2a\sqrt{10}$.
9. (i) $\sqrt{185}$ feet; (ii) $\sqrt{(a^2 + b^2 + c^2)}$;
(iii) $\sqrt{\{a^2 + (b - h)^2 + c^2\}}$.
10. $\sqrt{\{p^2 + q^2 + (b - a)^2\}}$. 11. (i) $\frac{r}{2}(2 - \sqrt{3})$; (ii) $\frac{r}{2}(2 + \sqrt{3})$.

EXERCISE IX.

See ch. VI., § 7.

Approximate equalities are indicated by the sign + or − following the left hand expression. The pupil may use the sign $\doteqdot$ as instructed on p. 49.

A.

3. (i) 20 sq. inches, 4 sq. inches;
(ii) 1·44 cms.^{2}, 0·16 cm.^{2}; (iii) 9·8 cms.^{2}; 0·49 cm.^{2}.
4. Fraction of area = (fr. of side)^2.
5. 1/144. 6. 1/400. 7. $A = a^2 + 2ab + = a(a + 2b) +$.
8. (i) $A = 40 \times 46 = 1840$ feet^2; (ii) $53 \times 60 = 3180$ feet^2;
(iii) 2175 feet^2.
9. (i) 3000 feet^2, $(5/50)^2 = 1/100$; (ii) 6248, 1/484.
10. (i) $W = aw(a + 4b)$; (ii) $400b^2/(a + b)^2$.
11. $314\frac{1}{2}$ sq. inches. 12. (i) $I = 2ab + b^2$; (ii) $I = 2ab +$.
13. (i) 22·8 cms.^{2}; (ii) 1/1600;
(iii) $b^2/b(2a + b) = b/(2a + b) = 1/81$; (iv) the latter.
17. $A = 2\pi rt +$. 18. (i) $A = 4\pi r(r + 2t) +$; (ii) $8\pi rt +$.
20. Expr. = (i) $a^2 + 4ab + 46^2$; (ii) $9p^2 + 6pq + q^2$;
(iii) $p^2 + \frac{1}{2}pq + \frac{1}{16}q^2$; (iv) $a^2 + 2\sqrt{5}\,.\,b + 5b^2$;
(v) $1 + p + \frac{1}{4}p^2$; (vi) $1 + 0{\cdot}006t + 0{\cdot}000009t^2$;
(vii) $1 + 2ct + c^2t^2$; (viii) $a^2 + 2pab + p^2b^2$;
(ix) $l^2(a^2 + 2ab + b^2)$; (x) $r^2(1 + \frac{1}{2}pt + \frac{1}{16}p^2t^2)$.
21. (i) $l = 1 + ct$; (ii) $l_t = 4(1 + ct)$; (iii) $l_t = l_0(1 + ct)$.
22. (i) $A_t = (1 + ct)^2 = 1 + 2ct +$; (ii) $A_t = l^2(1 + 2ct) +$;
(iii) $A_t = A_0(1 + 2ct) +$.
23. (i) $I_t = 2\pi rct +$; (ii) 0·4092 sq. cm.

B.

24. (i) 4·5 cms. −; (ii) 0·25 in excess.
25. $\sqrt{10} = \sqrt{9 + 1} = 3 + \dfrac{1}{2 \times 3} = 3\frac{1}{6}$. The other roots are:
$4\frac{1}{4}$, $7\frac{3}{7}$, $10\frac{1}{2}$, $13\frac{3}{13}$, $5\frac{1}{5}$, $20\frac{1}{4}$, 4·2, 9·13.
26. (i) 2·035 sq. cms.; (ii) 3·6 sq. inches; (iii) 6·3 feet.
27. Expr. = (i) $a + p/2a$ −; (ii) $a + p/a$ −; (iii) $a + b^2/2a$ −;
(iv) $r + 1/2r$ −; (v) $2p + q/p$ −; (vi) $4 + a/8$ −;
(vii) $9 + nt/18$ −.
28. (i) $\sqrt{13} + a/\sqrt{13}$ − $= 3{\cdot}6(1 + a/13)$ −; (ii) $2{\cdot}44(2 + a/24)$ −;
(iii) $\sqrt{a} + b/2\sqrt{a}$ −; (iv) $\sqrt{p} + q/\sqrt{p}$ −; (v) $a + \sqrt{(b/a)}$ −;
(vi) $2\sqrt{p} + \frac{3}{4}\sqrt{(q/p)}$ −; (vii) $3\sqrt{a} + \sqrt{(b/a)}$ −;
(viii) $\frac{1}{8}\{\sqrt{a} + \frac{3}{2}\sqrt{(b/a)}$ −.
29. (i) $D = 1{\cdot}22\sqrt{(H + h)}$; (ii) $D = 1{\cdot}22\sqrt{(H + h/2\sqrt{H})}$ −.
30. (i) $D = 25$ miles; (ii) 0·61 mile; (iii) $d = 0{\cdot}61h/\sqrt{H}$ −.

EXERCISE X.

See ch. VI., § 7.

2. $A = a(a - 2b) +$. 3. $26 \cdot 4 \times 25 = 660$ sq. inches.
4. 916 sq. cms.
6. Exp. = (i) $a^2 - 4ab + 4b^2$; (ii) $\frac{a^2}{4} - ab + b^2$;
(iii) $p^4 - 2p^2q^2 + q^4$; (iv) $1 - 2p^2 + p^4$;
(v) $1 - 4p^2 + p^4$; (vi) $1 - 2ct + c^2t^2$;
(vii) $A(1 - 2ct + c^2t^2)$; (viii) $r^2(1 - 2ct + c^2t^2)$;
(ix) $a^2(1 - 2bt^2 + b^2t^4)$; (x) $r^4 - 2abr + \frac{1}{4}a^2b^2$.
7. (i) $V = 2\pi rtl$; (ii) $C = \pi rtlc/6v$.
8. (i) $A_t = A(1 - 2ct) +$; (ii) $d = 2ctA$.
9. (i) $A = \pi r(r + l)$; (ii) $d = 2\pi rct(r + l) -$.
11. (i) $3a - b/6a -$; (ii) $4 \cdot 3(p - q/6p) -$; (iii) $4 - 2 \cdot 1t^2/8 -$;
(iv) $a - b^2/2a -$; (v) $\sqrt{a} - 2\sqrt{(b/a)} -$.
12. (i) $h = a - \sqrt{(a^2 - b^2)} = b^2/2a +$; (ii) 1/3 inch.

EXERCISE XI.

See ch. VI., § 7.

A.

1. $a^2(a + 3b) +$. 2. $V = 10^2 \times \frac{3}{12} + = 25$ cu. inches.
3. $V = 1 + 3ct +$. 4. $I = 4\pi r^2ct +$.
6. Expr. = (i) $a^2(a + 6b) +$; (ii) $4a^2(2a + 3b) +$;
(iii) $(1 + p) +$; (iv) $a^2(a + 3pt) +$.
8. See ch. VI., § 8.
9. Expr. = (i) $a^2(a - 3b/2) -$; (ii) $\frac{p^2}{4}\left(\frac{p}{2} - 3q\right) -$;
(iii) $(1 - 6 \cdot 9k) -$; (iv) $a^2(a - 3nt/2) -$.
10. See ch. VI., § 8.
11. Expr. = (i) $a + d/3a^2 -$; (ii) $a - d/3a^2 -$.
12. $\sqrt[3]{10} = \sqrt[3]{(2^3 + 2)} = 2 + 2/12 - = 2\frac{1}{6} -$. The other roots are approximately $3\frac{1}{9}$, $3\frac{11}{12}$, $10 \cdot 1$, $6\frac{6}{7}$, $0 \cdot 8085$; $\cdot 289$; $1 \cdot 1875$.

B.

13. $V =$ (i) $1 + 3ct +$; (ii) $1 - 3ct +$.
14. $I = 4\pi r^3ct +$. 15. (i) $1 \cdot 004 +$ sq. inch; (ii) $1 \cdot 006 +$ cu. inch.
16. $V = 1 - 0 \cdot 009$ $\therefore$ (i) $l = 1 - 0 \cdot 003 = 0 \cdot 997$ inch;
(ii) $A = 1 - 0 \cdot 006 = 0 \cdot 994$ sq. inch.
17. (i) $L = L_0(1 + 0 \cdot 00006t)$; (ii) $A = A_0(1 + 0 \cdot 00012t)$.
18. $2 \cdot 88$, $5 \cdot 13$, $4 \cdot 32$, $6 \cdot 84$, $2 \cdot 4624$, $2 \cdot 9241$, $4 \cdot 9248$, $3 \cdot 545856$.
19. $2\sqrt[3]{2}$, $3\sqrt[3]{4}$, $6\sqrt[3]{10}$, $10\sqrt[3]{9}$, $2\sqrt[3]{0 \cdot 2}$, $6\sqrt[3]{0 \cdot 003}$, $\frac{1}{2}\sqrt{0 \cdot 7}$.
20. $a\sqrt[3]{b}$, $b\sqrt[3]{a}$, $b\sqrt[3]{a^2}$, $a\sqrt[3]{ab}$, $b\sqrt[3]{ab}$, $b\sqrt[3]{a^2b}$, $c\sqrt[3]{a^2b^2}$, $ab\sqrt[3]{c^2}$, $ab\sqrt[3]{a^2c}$, $bc\sqrt[3]{a^2b^2c}$, $3p\sqrt[3]{r}$, $3n\sqrt[3]{4m}$, $2\sqrt[3]{(p - q)}$, $2\sqrt[3]{2(a^2 - b^2)}$, $4p\sqrt[3]{(1 - 8t)}$, $a\sqrt[3]{(ab + 1)}$, $p\sqrt[3]{(1 - q^3)}$, $ab\sqrt{(a - b)}$.

EXERCISE XII.

See ch. VI., § 8; ch. IX., A.

The subject of a formula, where obvious, is sometimes omitted.

1. (i) $R = pq/(p + q)$; (ii) $pq/(q - p)$; (iii) $pq/(3p + 2q)$; (iv) $pq/(3q - 5p)$; (v) $q/(q - p)$; (vi) $q/(p - 2q)$; (vii) $q/(p + qr)$; (viii) $p/(pr - q)$; (ix) $qr/(pr + q)$; (x) $qr/(2pr - 3q)$; (xi) $r/(2q - 3pr)$; (xii) $r/(1 - pqr)$.
2. (i) $A = 2(b - 3a)/ab$; (ii) $A = c/a(b + c)$; (iii) $P = p(r - q)/r$; (iv) $D = d_3/d_1(d_2 + d_3)$; (v) $B = b/a(b + 1)$; (vi) $R = r_2/r_1(1 - 2r_2)$; (vii) $R = 3r_2/r_1(2 - 3r_2)$; (viii) $A = (a^2 + 2)b/a$; (ix) $A = b/a(1 - 3b^2)$; (x) $A = (a + b)(a - b)/ab$; (xi) $V = (2u + v)(2u - v)/2uv$; (xii) $R = pq/(2p + 3q)(2p - 3q)$.
3. (i) $P = (q + 1)/pq$; (ii) $(2 - 3q)/pq$; (iii) $(p + q - 1)/pq$; (iv) $pq/(p + q - 2)$; (v) $pq/(bp + aq - c)$.
4. (i) $V = uv/(u + v)$; (ii) $2uv/(v - u)$; (iii) $uvw/(v - u)$; (iv) $4au^2/(2u + v)(2u - v)$; (v) $av^2/(u + bv)(u - bv)$.
5. (i) $V = u + v$; (ii) $u - v$; (iii) $(v - u)/u$; (iv) $(u + v)(u - v)/uv$; (v) uv; (vi) u; (vii) u; (viii) $1/uv$; (ix) $(u - v)/u$; (x) $(u - v)/u^2$; (xi) $v^2/(u + v)$; (xii) $uv/(u + v)(u - v)$.
6. (i) $(a + b)/a^2b$; (ii) $(a - b)/ab^2$; (iii) $(b - a)/a^2b^2$; (iv) $(a + b + c)/abc$; (v) $(a^2 + b^2 + c^2)/abc$; (vi) $(a + c)(a - c)/abc$; (vii) $(a + b)^2/a^2b^2$; (viii) $(b - a)^2/a^2b^2$.
7. (i) $(2 - 3p)/6p^2$; (ii) $a^2 + b^2)/pab$; (iii) $(m + n)(m - n)/amn$; (iv) $(pa + qb)^2/p^2q^2a^2b^2$ or $\{(pa + qb)/pqab\}^2$; (v) $(4p^3 - 9q^3)/6p^2q^2$.
8. (i) $l = (A_1b_2 + A_2b_1)/b_1b_2$; (ii) $(nb_2 + b_1)A/b_1b_2$.
9. (i) $T = mb/(m + b)$; (ii) $mb/(2m + 3b)$; (iii) $nm/(n + 1)$; (iv) $b/(n + 1)$; (v) $nm/(pn + q)$ and $b/(pn + q)$.
10. (i) $n = aq/bp$; (ii) $aq/3bp$; (iii) $9aq/bp$; (iv) aqs/bpt; (v) aqs^2/bpt^2.
11. (i) $V = (aq + bp)t/pq$; (ii) $n(aq + bp)/p$; (iii) $T = Qp/a$; (iv) $T = Qpq/(aq + bp)$; (v) $T = (Q - Q_0)q/b$; (vi) $(Q - Q_0)pq/(aq + bp)$.
12. $(p^2 + q^2)/p^2q^2$.
13. $(p + q)(p - q)/p^2q^2$.
14. (i) $d = 2\pi n(pR - Pr)/Pp$; (ii) $t = p/P$.
15. (i) $w = 300(mq - np)/pq$; (ii) $d = 200\pi(mq - np)/pq$; (iii) $d = 250\pi(mq - np)/11pq$; (iv) $A = 90000\pi(mq + np)(mq - np)/p^2q^2$; (v) $C = 125\pi c(mq + np)(mq - np)/3p^2q^2$.
16. (i) $d = V(A_2 - A_1)/A_1A_2$; (ii) $V(r_2 + r_1)(r_2 - r_1)/\pi r^2{}_1r^2{}_2$.
17. (i) $(p - 1)/p$; (ii) $h = (p - 1)V/pA$; (iii) $(p - 1)V/\pi pr^2$; (iv) $(p - q)V/pA$ and $(p - q)V/\pi pr^2$.
18. (i) $v = (p - q)^2V/p^2$; (ii) $d = (2p - q)qV/p^2A$.
19. $d = V\{ap^2 - A(p - q)^2\}/aAp^2$.
20. (i) $n = (p - q)^2A/ap^2$; (ii) $\{(p - q)R/pr\}^2$.

EXERCISE XIII.

See ch. VI., § 8; ch. IX., B.

A.

1. (i) $9 - a$; (ii) $23 - a$; (iii) $a - 5$; (iv) $2(p + 4)$; (v) $8(p - 1)$; (vi) $42 - r$; (vii) $6 + r$; (viii) $23r + 18$; (ix) $6(5r - 3)$; (x) $p + 8b$; (xi) $(2a + p)(2a - p)$; (xii) $(3m + 2p)(3m - 2p)$; (xiii) $3r - 5$; (xiv) $20 - 2p$.
3. (i) $a \not> 9$; (ii) $a \not> 23$; (vi) $r \not> 42$; (xi) $p \not> 2a$; (xiii) $r \not< 5/3$.
4. (i) $3p$; (ii) $4 + t$; (iii) $2(t - 1)$; (iv) $5a$; (v) $a - 12b$; (vi) $2(3q - p)$; (vii) $23n - 10m$; (viii) $a(p + 4)$; (ix) $2a(2 - p)$; (x) $4a + b^2$; (xi) $p - 2\sqrt{2}$; (xii) $2\sqrt{2}$.
5. (i) $a \not< 12b$ or $a/b \not< 12$; (ii) $p \not> 3q$; (iii) $m \not> 2{\cdot}3n$.
7. (i) $t = 800/q(q - 8)$; (ii) $t = Qd/q(q - d)$.
8. $72d/(14 - d)$. 9. $720/p(p + 1)$.
10. $90(8 - p)/p(p + 1)$.
11. (i) $31{\cdot}5/a(a + 6{\cdot}3)$; (ii) $1{\cdot}2/(8{\cdot}4 - p)$; (iii) $(3{\cdot}7 - p)/(p + 6{\cdot}1)$; (iv) $2(11 - p)/(p - 6{\cdot}1)$; (v) $3(5{\cdot}2 + a)/(3 + a)$; (vi) $3(5{\cdot}2 + a)/(3 - a)$; (vii) $a(0{\cdot}7a + 1)/(0{\cdot}7a + 3)$; (viii) $b(3{\cdot}1b - 4)/(7 - 3{\cdot}1b)$; (ix) $5(2a - 1)/2{\cdot}3a(7{\cdot}7a - 5)$; (x) $8(96 - 7p)/3(32 - p)$; (xi) $a/(12 + a)$; (xii) $(p - 21)/(30 - p)$.
12. (i) $2b/a(a + 2b)$; (ii) $q/3p(3p - q)$; (iii) $b/5a(5a + b)$; (iv) $pq/3(3 - p)$; (v) $pn/(m - n)$.
13. $lps/(36 - s)$. 14. (i) $clt/(1 + ct)$; (ii) $clt/(1 - ct)$.

B.

17. (i) $1 + a +$. (ii) $1 - a +$.
18. (i) $1 + pt +$; (ii) $d(1 - pt) +$; (iii) $d(1 - 0{\cdot}0006t) +$; (iv) $1 + \frac{b}{a} +$; (v) $a + b +$; (vi) $(a - b)/a^2 +$; (vii) $(a - b)/a^2 +$; (viii) $(a - b)/a^2 +$; (ix) $(p + q)r/p^2 +$; (x) $a(p^2 - q^2)/p^4 +$.
19. (i) $18d(14 + d)/49 +$; (ii) $clt(1 - ct)t$, $clt(1 + ct) +$.
20. £$320{,}000(1/80 - 1/80\frac{1}{2})$ = £$4000\,\{1 - 1/(1 + \frac{1}{160})\}$ = £$4000 \times 1/160$ = £25 – .
21. £12 5s. ×. 22. $211200Lc/l$.
23. $2md/s^2$.
24. (i) $(a - b)t$; (ii) $(d/c^2 - b(a^2)t$; (iii) $1 - 2a$; (iv) $1 + 2a$; (v) $(1 - 2pt)/a^2$; (vi) $(a - 2b)/a$; (vii) $(p + 2q)/p^3$; (viii) $2q/p^3$; (ix) $1 - a/2$; (x) $(2a^2 + p)/2a^3$; (xi) $r/2p^2$; (xii) $ct/2000$; (xiii) $2ct/l^3$; (xiv) $1 + b/2a$; (xv) $q/2p\sqrt{p}$; (xvi) $1 + 3a$; (xvii) $(1 - 3ct)/r^3$; (xviii) $1 + 3b/a$; (xix) $1 + p/3$; (xx) $r/3a^4$.
25. $4v/\pi a$. 26. $4v/\pi a + 2v/\pi r$. = .

27. $9h/D\sqrt{D}$; 1/81 lb. 28. $9h/D\sqrt{D}$.
29. $hT/2{\cdot}21\sqrt{l}$. 30. The same.
31. $1 + a + a^2$. 32. $1 - a + a^2$.
33. (i) $\frac{1}{a}\left(1 + \frac{b}{a} + \frac{b^2}{a^2}\right)$; (ii) $\frac{1}{a}\left(1 - \frac{b}{a} + \frac{b^2}{a^2}\right)$;
(iii) $1 - ct + c^2t^2$.

C.

34. P = (i) $(2p + 5)/(p + 2)(p + 3)$; (ii) $1/(p + 2)(p + 3)$;
(iii) $(2p + a + b)/(p + a)(p + b)$; (iv) $(p + a)(p + b)/(a - b)$;
(v) $(p - a)(p + a)/2p$; (vi) $(2p - a)(2p + a)/2a$;
Q = (vii) $(m + n - 2a)/(m - a)(n - a)$; (viii) $(m - n)/(m - a)(n - a)$;
(ix) $(pm + a)(qn + b)/(pm - qn + a - b)$;
(x) $(p - n)(q - m)/(pm - qn)$; (xi) $(p - a)(q - a)/(p - q)(p + q - a)$;
M = (xii) $(13m - 12n)a/(2m - 3n)(3m - 2n)$;
(xiii) $5ma/(2m - 3n)(3m - 2n)$; (xiv) $(1 - pa)(1 - qh)/(pa - qh)$.
35. (i) $(a + b + 1)/(a + b)^2$; (ii) $b/(a + b)^2$; (iii) $b/(a - b)^2$;
(iv) $a/(a - b)^2$; (v) $(a + b)/(a - b)^2$; (vi) $(a - b + 1)/(a^2 - b^2)$;
(vii) $(a + 2b)/(a^2 - b^2)$; (viii) $a/(a^2 - b^2)$; (ix) $(2a + 1)/(a^2 - b^2)$;
(x) $(2b + 1)/(a^2 - b^2)$; (xi) $2/(a - b)$; (xii) $2/(a - b)$;
(xiii) 0. (xiv) 0; (xv) $2p/(p - q)(p + q)^2$;
(xvi) $2q/(p + q)^2(p - q)$; (xvii) $2q/(p - q)^2(p + q)$;
(xviii) $1/(p - q)$; (xix) $(p + q/(p - q)^2$;
(xx) $(p^2 + q^2)/(p - q)(p + q)^2$.
36. (i) $2a^2/(a + b)^2(a - b)$; (ii) $2a^2/(a + b)(a - b)^2$;
(iii) $2ab/(a + b)^2(a - b)$; (iv) $-2ab/(a + b)(a - b)^2$;
(v) $b/(a + b)^3$; (vi) $(a^2 + b^2)/(a + b)^3$; (vii) $2a^2/(a - b)^3$;
(viii) $2b^2/(a + b)^3$.

EXERCISE XIV.

Ch. VI., § 9; ch. X. Note especially in the "literal" examples that the systematic application of the rules of ch. X. leads directly to the simplest form of the answer. In Nos. 7 and 14 (and, if necessary) in No. 23 the working may be an actual transcription of the pupil's solution of the arithmetical case (as in ch. X., § 3).

A.

1. 2·3. 2. 3. 12·7. 4. 5. 17·28. 6. 27·5.
7. (i) $n = (c - b)/a$; (ii) $(b + c)/a$;
(iii) $(n + a)b = c$; $n + a = c/b$; $n = c/b - a$; (iv) $a + c/b$;
(v) $a + bc$; (vi) $(c - b)a$.
8. 3. 9. 6. 10. 23. 11. 8. 12. 5. 13. 7.
14. (i) $n = (d/c - b)/a$; (ii) $a + c/b$; (iii) $(b + cd)/a$;
(iv) $(cd - b)/a$;
(v) $b/(n + a) = c$, $b = (n + a)c$, $n + a = b/c$, $n = b/c - a$;
(vi) $(b + a/d)/c$.

15. 4. 16. 2·7. 17. 1·9. 18. 3. 19. 9. 20. 4·5. 21. 3. 22. 3.
23. (i) $n = (a/d + c)/b$; (ii) $n = \{c(d + e) - b\}/a$;
(iii) $n = \{a/(e - d) - c\}/b$.
24. $n = \{c(d - e) + b\}/a$.
25. $n = \{(a + b)r - q\}/q$.
26. $n = \{(p + q)a + c\}/b$.
27. $n = \{a/b(c + 1) - q\}/p$.
28. $n = \{a/(b - c) + q\}/p$.
29. $n = a(1 + q + b/p)$.
30. $n = a(b - q)/p$.
31. $n = p\{a/(b + c) - q\}$.

B.

32. (i) $n = (W - b)/m$; (ii) $m = (W - b)/n$.
33. (i) $t_1 = (l - n_2t_2)/n_1$; (ii) $t_2 = (l - n_1t_1)/n_2$.
34. $t = (S - S_0)/i$.
35. (i) $S_0 = S + rt$; (ii) $t_2 = (S_0 - S)/r$.
36. (i) $l = kbd^2/W$; (ii) $b = Wl/kd^2$; (iii) 6·8 feet.
37. $P = 21{,}000\ H/nlD^2$; $P = 35$ lb.
38. $d = \sqrt{\{L/7{\cdot}11\}}$; 1·2 inches nearly.
39. $L = \sqrt[3]{(10p)}$.
40. $l = \sqrt{(8ds/w)}$; 106 feet 8 inches.
41. (i) $h = d^2/1{\cdot}49$; (ii) $d = W^2/20A^2$; (iii) $D = 324/P^2$.
42. $i = (s - s_0)/t$.
43. (i) $w_1 = ns + w_2$; (i.) $n = (w_1 - w_2)/s$.
44. (i) $d_0 = d - (s_1 - s_2)t$; (ii) $t = (d - d_0)/(s_1 - s_2)$;
(iii) $s_1 = (d - d_0)/t + s_2$.
45. $R = 12r(d_2 - d_1)$. 46. The same.
47. (i) $T = it/(d_2 - d_1)$; (ii) $t = T(d_2 - d_1)/i$;
(iii) $d_2 = it/T + d_1$.
48. $h = Ld(W + P)/W$.
49. $W = (D - 0{\cdot}2)^2/0{\cdot}45$.

EXERCISE XV.

See ch. VI., § 9; ch. X.

A.

1. 3·4. 2. 34·5. 3. 33. 4. 169. 5. 257·4.
6. 5·2. 7. $6\frac{1}{3}$. 8. (i) $n = (a - c)/b$; (ii) $\{b - a/(e - d)\}/c$;
(iii) $\{c - b/(a - e)\}/d$; (iv) $b(a - c)$; (v) $a(b + c)$;
(vi) $(b - a + d)/c$; (vii) $\{c - (a - e)/b\}d$.
9. $n = \{q - b/(a - c)\}/p$.
10. $n = b - 1/p(a - c)$.
11. $n = \{b - (a - q)/p\}/c$.
12. $n = \{p - (c^2 - a^2)/b\}/q^2$.
13. $n = (a - qr^2/p)/b^2$.
14. $b = d - (P_0 - P)/t$.
15. $s_2 = s_1 - (d_0 - d)/t$.

16. $d_2 = d_1 - (d_1 - d)t/T = \{d_1(T - t) - dt\}/T$.
17. $d_2 = d_1 - d_1t/T = d_1(T - t)/T$.
18. $T = 24 - 2R = 2(12 - R)$.
19. $d_2 = d_1 - D(h - E)/(H - h) = 11$ feet nearly.
20. $t = (12p - 37{\cdot}7nD)/3{\cdot}15n^2$.
22. $l_1 = \frac{1}{2}(R_1 + R_2 + R_3) - E$.
23. $d = \sqrt[3]{\{D^3 - 12I/B\}}$.

B.

24. 7·15. 25. 4.
26. (i) 3; (ii) 3; (iii) 5; (iv) 2·1.
27. (i) $n = (c - b)/(a - 1)$; (ii) $(b + c)/(a + d)$;
(iii) $(a - c)/(b - d)$; (iv) $(a + d)/(c - b)$.
28. 150. 29. 45 and 15. 30. £50, £70. 31. 69.
32. (i) 12; (ii) 13; (iii) $12\frac{1}{2}$; (iv) 4·1; (v) 11; (vi) 6; (vii) 8; (viii) 4.
33. (i) $n = bc/(a - b)$; (iii) $(ab + cd)/(c - a)$;
(v) $(ab + cd)/(a + c)$; (vii) $(ac + df)/(de - ab)$.
34. 29, 32 or 104. Draw diagram. C may be (i) between A and B, (ii) below A or (iii) above B.
(i) requires $3(41 - n) = 4(n - 20)$ or $4(41 - n) = 3(n - 20)$;
(ii) requires $4(20 - n) = 3(41 - n)$ which gives no result;
(iii) requires $4(n - 41) = 3(n - 20)$.
35. Four positions possible :—
(i) between trees 36 yards from first;
(ii) outside trees 120 yards from first;
(iii) and (iv) the same positions with respect to the second tree.
36. 60. 37. (i) 22; (ii) 12; (iii) 2·5; (iv) 9.
38. (i) $n = (ad + bc)/(b + d)$; (iv) $(af + ce)/(bf + cd)$.
39. £408.
40. (i) $12\frac{1}{8}$; (ii) 4; (iii) 9; (iv) 5; (v) 8; (vi) 7; (vii) 12; (viii) 3; (ix) 8·23; (x) 5.
41. 11 mls./hr.
42. (i) 49; (ii) 2; (iii) 2·79; (iv) 9·35; (v) 7·3; (vi) 5·5; (vii) $\frac{3}{4}$; (viii) 2.
43. 30. 44. 60.
45. (i) 48; (ii) $4\frac{16}{17}$; (iii) $\frac{15}{19}$; (iv) $1\frac{7}{8}$; (v) 16; (vi) 4; (vii) 7; (viii) 5·6; (ix) $5\frac{3}{11}$; (x) $2\frac{6}{11}$.
46. (i) $n = pc/(pb - qa)$; (ii) $(ap + bq)/(aq + bp)$;
(iii) $(c - aq - bs)/(ap - br)$;
(iv) $(a - b)/c(a + b)$; (v) $c/(a - b)p + q/p$;
(vi) $\left(\frac{a}{p} + \frac{b}{q} - \frac{c}{r}\right)\Big/\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\right)$.

C.

47. $p = q(an + b)/(a - bn)$; $q = p(a - bn)/(an + b)$.
48. $a/b = (1 + cn)/(1 - cn)$. 49. $a - b = c/(pn - q)$.
50. $h = (c - c_2)/(2c_2 + c)$. 51. $V_2 = V_1 - il/cS$.

52. $h = a - F/4\pi I$; $a = 4\pi Ih/(4\pi I - F)$.
53. $n = \sqrt{(2b^2 - Rbc/Ld)}$.
54. $V_1 = V - 4\pi d_1d_2Q/S_1(d_1 + d_2)$.
55. $q = 1/(1 - d) - a_0/a$.
56. $r_2 = r_1 - r_1RI/e = r_1(1 - RI/e)$.
57. $V_1 = V(1 - RCl/t)$. 58. $r = \sqrt{\{B/(A - e)\}}$.
59. $u = \sqrt{(v^2 - 2Fsg/W)}$. 60. $r = \sqrt{\{a^2 - 4ev/(p_1 - p_2)\}}$.

EXERCISE XVI.

See ch. VI., § 10.

A.

1. (i) $S = S_0 - mt$; (ii) $T = (S_0 - mt)/m$.
2. (i) $s = \sqrt{lb}$; (ii) $p = l - \sqrt{lb}$; (iii) $q = \sqrt{lb} - b$.
3. (i) $C = \pi d$; (ii) $C = 2\pi r$; (iii) $r = C/2\pi$; (iv) $r = l/2\pi n$; (v) $n = l/2\pi r$; (vi) $A = \pi r^2$; (vii) $r = \sqrt{(A/\pi)}$; (viii) $V = \pi r^2h$; (ix) $h = V/\pi r^2$; (x) $r = \sqrt{(V/\pi h)}$.
4. (i) $h = V/lb$; (ii) $h = V/\pi r^2$; (iii) $h = 0{\cdot}16V/lb$; (iv) $h = 0{\cdot}16V/\pi r^2$.
5. (i) $d = (s_1 - s_2)t$; (ii) $d_2 = d_1 - (s_1 - s_2)t$; (iii) $D_1 = (s_1 - s_2)d_1/s_1$; (iv) $D_2 = (s_1 - s_2)d_2s_2$.
6. (i) $d = d_0 + (s_1 - s_2)t$; (iii) $D_1 = d_0 + (s_1 - s_2)d_1/s_1$; (iv) $D_2 = d_0 + (s_1 - s_2)d_2/s_2$.
7. (i) $m = (s_1 - s_2)t$; (ii) $d = d_0 - (s_1 - s_2)t$; (iii) $T = d_0/(s_1 - s_2)$; (iv) $L = d_0s_1/(s_1 - s_2)$; (v) $t = M/(s_1 - s_2)$; (vi) $t = (d_0 - D)/(s_1 - s_2)$; (vii) $L_1 = s_1L_2/s_2$; (viii) $L_2 = s_2L_1/s_1$; (ix) $L_2 = s_2(d_0 - d)/(s_1 - s_2) + d_0$; (x) $L_1 = s_1(d_0 - d)/(s_1 - s_2)$.
8. (i) $L = lh/w$; (ii) $L = 12lh/w$; (iii) $L = 4lh/w$; (iv) $L = 4lb/21$; (v) $L = lb/63$; (vi) $C = plb/63$; (vii) $C = plb/756$.
9. (i) $h = nv/A$; (ii) $d_n = d_0 + nv/A$; (iii) $n = (d_n - d_0)A/v$; (iv) $n = (l - b)A/v + 1$.
10. (i) $W = \pi r^2hc$; (ii) $W = w + \pi r^2dc$; (iii) $c = (W - w)/\pi r^2d$; (iv) $C = P/\pi r^2d$.

B.

11. h = (i) 6 in. ; (ii) 5·4 in. ; (iii) 1·92 in.
12. R = (i) 1050 ; (ii) 470·4. 13. H = (i) 176 ; (ii) 410·6.
14. d = 3·41 in. 15. d = 3·513 in. 16. 19·6 lbs.
17. D = 17·736 in. 18. s = (i) 26·22 ; (ii) 37·59.
19. w = (i) 186,000 ; (ii) 134,560 ; (iii) 76,230.
20. (i) £5 11s. ; (ii) £9 6s. 9d. ; (iii) £19 10s. ; (iv) £67 10s. ; (v) £108 15s. ; (vi) £529 3s. 4d.

C.

21. 14 and 26, etc., for $1426 = 14 \times 100 + 26$.
22. $5(20m + 12) + d = 100m + 60 + d = 578 \therefore 100m + d = 518$. Birthday is 18th day of 5th month. Rule: Subtract 60; last two figures give the day, the rest the month.
23. $10\{10(C - 4) + 4\} + n = 100\,C + n - 400 + 40$. Add 400 to 1289, take 40; answer, 1649.
24. Subtract 200 and reject the final 33; answer, 265.
25. Subtract 250; last digit = w, next = l, others = p; 7, 4, 124.
27. $N = 10a + b = 9a + (a + b)$. 9a is divisible by 9, hence if $(a + b)$ is also divisible the number is divisible, etc.
28. $N = 100a + 10b + c = 99a + 9b + (a + b + c)$. The first two terms are divisible, therefore $a + b + c$ is divisible, etc.
29. Yes. 30. Apply No. 28.
31. $N = 100a + 10b + c = 99a + (10b + c + a)$. 99a is divisible by 11, hence if N is divisible $(10b + c + a)$ must be divisible.
32. $N = 1000a + 100b + 10c + d$
$= 990a + 99b + \{(10c + d) + (10a + b)\}$.
Therefore, etc., as in No. 31.
34. $N = 1000a + 100b + 10b + a = 1001a + 110b = 11(91a + 10b)$.
36. $N = 99a + 11b + \{(a + c) - b\}$. Therefore last term must be zero or divisible by 11.
38. 102036 and 151602 are divisible.
39. (i) $(b - a)(1 - t)$; (ii) $(a - c)(t + 1)(t - 1)$.
41. 9. 42. 594 is the difference.
43. (i) 2 and 9; (ii) 9 and 0; (iii) 9 and 9.
44. (i) $(a - d)(t - 1)(t^2 + t + 1) + (b - c)(t - 1)t$;
(ii) $(a - e)(t - 1)(t + 1)(t^2 + 1) + (b - d)(t - 1)(t + 1)t$.
46. 8. 47. 9 or 0.

D.

51. (i) 8 feet 5 inches; (ii) $67\frac{3}{4}$, $74\frac{3}{4}$ miles;
(iii) $82\frac{1}{2}$ miles; (iv) about 30 feet.
52. (i) June 21st, 18 hours $34\frac{1}{2}$ minutes; (ii) 55 days.
54. (i) 51·5°; (ii) 27°; (iii) between 4·5 and 5·5 minutes.
55. (i) 800; (ii) 560; (iii) 460; (iv) 175.
60. (i) About 20 miles per hour;
(ii) after 2 minutes $7\frac{1}{2}$ seconds, at 650 yards from A;
(iii) after 1 minute 10 seconds, after nearly 5 minutes;
(iv) 440 yards from A, 390 yards from B.

E.

61. $hb(a + \frac{\pi}{4}b)$. 62. $\frac{4W}{hb(4a + \pi b)}$.

63. $b(3a^2 - \frac{1}{3}\pi b^2)$. 64. $\pi b^2\left\{\frac{a}{4} + \frac{b}{6}\right\}$.

65. $\pi b\{\frac{3}{4}a^2 - \frac{4}{3}b^2\}$. 66. $hb(3a - \pi b)$. 67. $whb(3a - \pi b)$.
68. (i) $2\pi r(r + h)$; (ii) $a(a + 2s)$; (iii) $\pi r\ (r + s)$.
69. $\pi b(a + b)$.
70. $a\{4(a + b) - \pi a\}$.
71. (i) $\pi(R - 3r)(R + 3r)$; (ii) $\pi(R - 4r)(R + 4r)$.
72. $\pi(R - 4r)(R + 4r)$.
73. (i) $\pi(a + Lb)(a - Lb)$; (ii) $\pi(a + Lb)(a - Lb)cw$.
74. (i) $4\pi\{b - a\}b$; (ii) $\pi w\ (w + 2r)$; (iii) $\pi w(2r - w)$.
75. (i) $2\pi L(R + r)$; (ii) $2\pi[L(R + r) + R^2 - 2rR + 3r^2]$.
76. (i) $2\pi L(2r + t)$; (ii) $2\pi[L(2r + t) + 2r^2 + t^2]$.
77. (i) $2\pi L(2r - t)$; (ii) $2\pi[L(2r - t) + 2r^2 - 4rt + 3t^2]$.
78. (i) 9 feet ; (ii) $d = \sqrt{(L - h)(L + h)}$.
79. $h = \sqrt{(L - d)(L + d)}$.
80. $\sqrt{(d + 3t)(d - 3t)}$.
81. (i) $a^2 + 5a + 6$; (ii) $a^2 + a - 6$;
(iii) $a^2 - 13a + 40$; (iv) $a^2 - 5a - 36$;
(v) $a^2 + 6a + 9$; (vi) $a^2 - 14a + 49$;
(vii) $6a^2 + 7a + 2$; (viii) $12a^2 - 25a + 12$;
(ix) $21a^2 + 5a - 50$; (x) $21a^2 - 2a - 40$;
(xi) $180a^2 + 81a - 77$; (xii) $4a^2 + 12a + 9$;
(xiii) $25a^2 - 60a + 36$; (xiv) $10a^4 - 21a^2 - 2$.
82. (i) $a^2 - 5ab - 14b^2$; (ii) $2a^2 - 9ab + 4b^2$;
(iii) $10a^2 + 29ab + 10b^2$; (iv) $49a^2 - 56ab + 16b^2$;
(v) $70a^2 + 2ab - 12b^2$; (vi) $26a^2 + 81ab - 35b^2$;
(vii) $a^3 + 6a^2b + 12ab^2 + 8b^3$;
(viii) $8a^3 - 12a^2b + 6ab^2 - b^3$;
(ix) $8a^3 - 36a^2b + 54ab^2 - 27b^3$;
(x) $64a^3 + 144a^2b + 108ab^2 + 27b^3$.
83. (i) $a + 4$; (ii) $a - 1$; (iii) $a + 4$;
(iv) $a + 2$; (v) $2a + 3$; (vi) $5a + 4$;
(vii) $2a - 3b$; (viii) $2a - 5b$; (ix) $3a + 7b$;
(x) $7a - 2b$.
84. (i) $(a - 3)(a - 2)$; (ii) $(p - 6)(p + 1)$;
(iii) $(p - 1)(p + 6)$; (iv) $(a - 12)(a + 11)$;
(v) $(a - 3)^2$; (vi) $(p - 7)^2$;
(vii) $(a - 4b)(a + b)$; (viii) $(4a - b)(a + b)$;
(ix) $(2a + 3b)^2$; (x) $(2a - 5b)(3a + b)$;
(xi) $(2a - 3b)(a + 4b)$; (xii) $(11a - 3)(5a + 2)$;
(xiii) $(a - 1)(b + 3)$; (xiv) $(b - 3p)(a + q)$;
(xv) $(p - 2q)(2a + 3b)$.
85. $ac + ad + bc + bd$. 86. $ac - ad - bc + bd$.
87. (i) $+ 9$; (ii) $+ 16$; (iii) $+ 36$;
(iv) $+ 121$; (v) $a + 4$; (vi) $- a + 16$;
(vii) $+ 29$; (viii) $+ 19$; (ix) $- 11p$;
(x) $- 17$; (xi) $+ 4b^2$; (xii) $- 11b^2$;
(xiii) $+ 4b^2$; (xiv) $+ 25q^2$; (xv) $+ 9n^2$;
(xvi) $+ 12b^2$; (xvii) $+ 52h^2$; (xviii) $- 11$;
(xix) $+ 3ab + 27b^2$ or $- 3ab + 12b^2$; (xx) $- a^2 + 5ab$.

88. (i) 2 ; (ii) 1 ; (iii) 13 ; (iv) 5 ;
(v) 12 or 2 ; (vi) 5·5 ; (vii) $\sqrt{3a^2 + 2a}$;
(viii) $\sqrt{7a^2 - 6a}$; (ix) $\sqrt{18a + 19}$; (x) 4·5.

F.

91. 1·48 miles.

92. (i) $2\pi\sqrt{\frac{l + h}{g}}$; (ii) $\frac{\pi h}{\sqrt{gl}}$.

95. 30·6 inches. 96. 18 yards.

97. About $1\frac{1}{2}$ miles. 98. ·07 second.

99. (i) 1,006,056 sq. miles ; (ii) 2,014,628,180 cu. miles.

G.

101. $m = S_0/(T + t)$. Monthly subscription = total amount to be raised divided by total time.

102. $b = (l - p)^2/l$. It states the breadth of an oblong of given length which suffers a given shortening on conversion into a square.

103. $s_1 = s_2d_1/(d_1 - D_1)$; $s_2 = s_1d_2/(d_2 + D_2)$.

104. $s_2 = s_1(L_2 - d_0)/(L_2 - d)$; $s_1 = L_1s_2/(L_1 - d_0 + d)$.

105. $v = \sqrt{(5hr)/2w}$; 30 mls./hr. 106. $H = nd^3/65$; 192.

107. (i) $D = 24\sqrt{(5d^3/Bl)}$; $d = \frac{1}{4}\sqrt[3]{(D^2l/45B)}$.

108. $s = 50\sqrt[3]{\{4/55(A + 0\cdot1s)\}}$; 10·8 knots. 109. $a = \sqrt{A} - s$.

110. 4 ft. 111. $a = \sqrt{(A + b^2)} - b$. 112. $a = \sqrt{(A + b)} + b$.

113. 5 ft. 114. $a = \sqrt{\{A + (l - b)^2/4\}} - (l + b)/2$.

115. 35·8 ft., 27·9 ft. 116. 5·7.

117. $b = \frac{2}{\pi}\left\{\sqrt{\pi(A + a^2)} - a\right\}$.

118. (i) 7, 1 ; (ii) 15 ; (iii) 6 ; (iv) 3·86 ; (v) 1·42 ; (vi) 16 ;
(vii) 7 ; (viii) 49·06, 2·93 ; (ix) 2·79 ; (x) 3·17.

119. (i) 6 ; (ii) 10 ; (iii) $\frac{1}{2}$; (iv) $\frac{1}{2}$; (v) 21 ; (vi) 5 ; (vii) 1 ;
(viii) 7 ; (ix) 0·3 ; (x) 2.

120. (i) $(c + d)/(a^2 - b^2)$; (ii) $a + b$; (iii) $2pq/(p + q)$;
(iv) $ab/(a^2 - b^2)(a - b)$; (v) $2c(a - b)/ab$; (vi) q ;
(vii) $q(p - q)/(p + q)$; (viii) $(a + b)/2$; (ix) .

EXERCISE XVII.

See ch. XI., § 1 ; ch. XII.

A.

1. $C = L/3$; $C = 3L/4$; $C = 5L/4$; 60 feet ; $26\frac{2}{3}$ feet.
2. (i) 3 ; (ii) 4·8 ; (iii) 6.
3. (i) $W = 21A/40$; (ii) $A = 40W/21$.
4. (i) $W = 13A/20$; $A = 20W/13$.
5. $v = 2\cdot5t$; 30 ft./sec.
6. $y = \frac{1}{4}x$. 7. $k = \frac{3}{2}$; $y = \frac{2}{3}x$.
8. $W = kV$; $W = 62\cdot5V$.
9. $W = 61\cdot1625V$.

B.

10. $Q = 24 + 3t/2$; $t = \frac{2}{3}(Q - 24) = \frac{2}{3}Q - 16$.
11. $Q = 24 - 3t/4$; $t = 32 - 4Q/3$; empty when $t = 32$ seconds.
12. $y = 24 + 3x/2$; $y = 24 - 3x/4$; $y = \frac{2}{3}x - 16$; $y = 32 - 4x/3$.
13. (i) $i = 2 \cdot 4W$; (ii) $L = 16 + 2 \cdot 4W$.
15. (i) $Q = 3t$; (ii) $Q = 21 + 5(t - 7) = 5t - 14$; (iii) 51 gallons.
16. If $t \not> 15$, $d = 2t/5$; if $t > 15$, $d = 1\frac{3}{4} + 17t/60$; 4 miles; 9·4 miles.
17. $t = 60d/17 - 6\frac{3}{17}$; 75 minutes.
18. (i) $r = (30 - 20)/5 = 2$; (ii) $Q_0 = 20 - 2 \times 3 = 14$; (iii) $Q = 14 + 2t$.
19. Note (using graph) that $(30 - 17)/(18 - 8) = 1 \cdot 3$ and
$17 - 1 \cdot 3 \times 8 = 6 \cdot 6$;
hence $y = 6 \cdot 6 + 1 \cdot 3x$.
20. $y = 64 - 3x$.
21. $0 \cdot 3t + 0 \cdot 4 \times 5 = 8$, $t = 20$ minutes.
22. (ii) $14 + 2t + 3(17 - t) = 52$, whence $t = 13$ minutes; (iii) $Q = 40 + 3t$; (iv) $1 + 3t$.
23. 1980; 180, 300; 6·6 minutes. If $t \not> 6 \cdot 6$, $n = 1980 - 180t$; if $8 \cdot 25 > t > 6 \cdot 6$, $n = 3960 - 480t$; 792; after 7 minutes.
24. When $x = 10$; $y = 5 + 8x$; $y = 135 - 5x$; when $x = 27$.

EXERCISE XVIII.

See ch. XI., § 2; ch. XIII., A.

A.

1. 89 feet. 2. (i) $40\frac{1}{2}°$; (ii) 23°. 3. 46·6 feet. 4. 26°.
5. (i) 76·1 cms.; (ii) 21·3 cms. 6. (i) $43\frac{1}{2}°$; (ii) 56°.
7. 181·9 feet; 42·9 feet. 8. 16·3 feet. 9. 691·5 feet.
10. 429 feet. 11. 173·3 feet; 26·7 feet.
12. 32·6 feet; 208·7 feet. 13. 41·1 feet.

B.

14. $a = p \tan \alpha$, $b = p \tan \beta$, $l = p (\tan \alpha + \tan \beta)$.
15. $p = l/(\tan \alpha + \tan \beta)$. 16. $p = l/(\tan \alpha - \tan \beta)$.
18. $H = h \tan \alpha/(\tan \alpha - \tan \beta)$. 19. $d = h (\tan \alpha - \tan \beta)$.
20. $h = d \tan \alpha \tan \beta/(\tan \alpha - \tan \beta)$. 21. 5016·7 yards.
22. 16,855 yards. 23. 1135·1 feet; 99·8 feet.
24. 128·2 feet; 170·1 feet. 25. 215·5 feet above eye.
26. 190·4 feet. 27. (i) $NA = d/\tan \alpha$; (ii) $h = d \tan \beta/\tan \alpha$.
28. 721·7 feet.

EXERCISE XIX.

See ch. XI., § 2: ch. XIII., B.

A.

1. 13·8 miles N., 5·9 miles E.; 12·6 miles N., 11·4 miles W.; 8·2 miles S., 20·4 miles W.; 34·7 S., 4·3 E.

2. (i) 72° E. of N. ; (ii) 24·7 miles.
3. 54° W. of S. ; 10·5 miles.
4. 14·5 miles ; 8·3 miles. 5. 20 miles ; 17·7 miles.
6. 7·4 miles N., 28·4 miles E. 7. 75° E. of N., 28·6 miles.
8. 19° E. of S., 43·3 miles.
9. Westing = 17 sin 40° + 6 sin 60° = 16·1 miles ;
northing = 17 cos 40° − 6 cos 60° = 10 miles ∴ tan a = 1·61 or course is 58° W. of N. Distance = 10/cos 58° = 18·9 miles.
10. Westing = 14 sin 40° − 6 sin 45° ;
southing = 6 cos 45° + 14 cos 40° ; d = 8 miles.
11. (i) b = c cos a ; (ii) a = c sin a ; (iii) c = b/cos a ;
(iv) c = a/sin a.
12. (i) a = b tan a ; (ii) b = a/tan a.
13. (i) cos a = b/c ; (ii) sin a = a/c ; (iii) tan a = a/b.

B.

14. 64°, 26° ; 65·6 yards.
15. AB = 42 cos 47° + 54 cos 35° = 72·9 yards.
16. 42·2 yards.
17. Draw perpendicular CD. BD = 180 cos 24° = 164·5. Hence AD = 115·5 = 150 cos BAC. (i) 7° E. of N.E., (ii) 53°.
18. 4·2 cms., 7 cms.
19. 9·1 inches, 11·7 inches.
20. 28·9 inches. See ch. v., § 12.
21. 18·6 cms. 22. See ch. v., § 12.
24. 29°, 18° ; 20·1 cms.
25. 5·8 inches.

C.

26. 5023 yards. 27. 4·3 miles.
28. 469·3 yards. See ch. v., § 12.
29. 689·1 yards. 30. 1·9 mile.
31. 9·5 miles, 12·5 miles. 32. 12·2 miles, 13·4 miles.
33. 2·7 miles. 34. half a mile.

EXERCISE XX.

See ch. xi., § 2 ; ch. xiv., A.

A.

1. 34° 6′ S. 2. 20° − 7° 56′ = 12° 4′.
3. 736 miles. 4. 6340 miles.
5. 2790 miles. 6. 21,600 × cos 60° ≐ 10,800 miles.
7. 3929 × cos 41° = 2966 miles.
8. 6018 cos 33° = 5049 miles.
9. 1765 miles S., 657 miles E.
10. 3155 miles S., 2716 miles W.

B.

11. 18·9 miles = 19′ N. ∴ lat. = 54° 19′ N.;
$14{\cdot}78/\cos 54° = 25'$ W. ∴ long. 34° 41′ W.
12. 40 miles S., 19 miles E., $25\frac{1}{2}°$ E. of S., 43·2 miles.
13. 28° 40′ E. of N., $66\frac{1}{2}$ miles.
14. Mid-lat. = 50° approx. Southing = 182 miles, easting = $300 \cos 50° = 193$ miles. Course 47° E. of S. (approx.), dist. = 267 miles.
15. $200 \times \cos 67° = 78$ miles = 1° 18′. Final lat. = 47° 5′ N.; mid-lat. = 47° 44′. $200 \times \sin 67° \div \cos 47\frac{2}{3}° = 274'$. Long. = 9° 3′ W.
16. 51° E. of S.; lat. 45° 45′ N., mid-lat. 47°, long. 20° 55′ W′.
17. 47° 30′ W. of N., 354 miles, mid-lat. 50°, long., 31° 44′ W.
18. 43° 20′ E. of N., 85° 38′ E.

EXERCISE XXI.

See ch. XI., § 2; ch. XIV., B.

A.

3. $\frac{12}{13}$. 4. No. 6. $\frac{12}{35}$. 8. $\frac{24}{7}$. 10. $\frac{84}{85}$. 12. $\frac{16}{65}$.

EXERCISE XXII.

See ch. XI., § 3; ch. XV.

A.

1. See ch. XV., A, § 5. (i) $y = a + bx$; (ii) $y = bx$; (iii) $y = a - bx$; (iv) $y = bx - a$.
2. (i) $y = 3 + 2x$; (ii) $y = 3x$; (iii) $y = 51 - 3x$; (iv) $y = 1{\cdot}1x - 2{\cdot}3$.
3. (i) $y = 19{\cdot}1 - 1{\cdot}7x$; (ii) $y = 3x - 13{\cdot}2$; (iii) $y = 0{\cdot}69 + 1{\cdot}3x$; (iv) $y = 2{\cdot}6x - 10{\cdot}7$.
4. $y = 1{\cdot}8 + 4{\cdot}8/x^2$; $y = 6{\cdot}6$; $y = 1{\cdot}848$.
5. $y = 5{\cdot}6\sqrt{x} - 2{\cdot}7$; 42·1.
6. $y = 1 + 3/(1 + x) + 5/(1 + x^2)$; $3\frac{1}{4}$.
7. Ch. XV., A, § 5.
8. $b = (Q - q)/(P - p)$; $a = (Pq - pQ)/(P - p)$.

B.

9. (i) $x = 2$, $y = 3$; (ii) $x = 1{\cdot}2$, $y = 7{\cdot}8$; (iii) $x = 10$, $y = 16{\cdot}6$; (iv) $x = 7{\cdot}8$, $y = 9{\cdot}2$; (v) $x = 24$, $y = 15$.
10. After $1\frac{1}{2}$ hours, $7\frac{1}{2}$ miles from Charing Cross.
11. In 4 days; 4 feet 8 inches.
12. 12·5 gms., 32 cms.

13. (i) $x = 4, y = 3$; (ii) $x = 13, y = 7$; (iii) $x = 5, y = 7$;
(iv) $x = 2{\cdot}1, y = 5{\cdot}3$; (v) $x = 9, y = 10$; (vi) $x = 2, y = 2$;
(vii) $x = 3, y = 5$; (viii) $x = 1\frac{1}{23}, y = 2$.
16. No. 17. No.
18. (i) $x = 3, y = 2$; (ii) $x = 1\frac{1}{2}, y = 2$;
(iii) $x = 1\frac{3}{7}, y = 3\frac{2}{11}$; (iv) $x = 3, y = 1$;
(v) $x = 5, y = 3$; (vi) $x = 12, y = 7$;
(vii) $x = 5, y = 2$; (viii) $x = 2\frac{1}{2}, y = 1$.

C.

19. (i) $5x - 2y = 1$; (ii) $14x - 15y = 1$;
(iii) $x - 8y = 1$; (iv) $\dfrac{4}{3x} + \dfrac{3}{2y} = \dfrac{1}{2}$ or $\dfrac{1}{9x} + \dfrac{1}{8y} = \dfrac{1}{24}$.
20. (i) $y = 4{\cdot}5, z = 6{\cdot}5$; (ii) $x = 5, z = 17$. Yes.
21. $x = 13, z = 14\frac{5}{8}$ from both.
22. $x^2 - 2xy \quad y^2 = 0$.
23. (i) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$; (ii) $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$;
(iii) $\dfrac{2x}{x^2 - y^2} = 1$ or $x^2 - y^2 - 2x = 0$.
24. $y = \frac{1}{3}$; $\cos a = \frac{1}{3}$, whence $a = 70\frac{1}{2}°$.
25. $2\sqrt{2}$. 26. $x + y = 1$.
27. $x = \frac{2}{3}$; $a = 30°$ approx.

EXERCISE XXIII.

See ch. XI., § 1; ch. XVI., A.

A.

2. (i) $k_1 = 800, k_2 = 200$; (ii) 160, 40;
(iii) 9600/7, 2400/7.
3. $xy = 29{\cdot}16$, $y = 1620$, $x = 145,800$.
4. $(5{\cdot}4, 5{\cdot}4)$; $5{\cdot}4\sqrt{2} = 7{\cdot}64$ approx.
5. (i) $x = 10, y = 20$; (ii) $x = 40, y = 20$.
7. $(p/q, pq)$.
10. $\sqrt{(a/b)}$.

B.

11. $(h - 1)A = 34{\cdot}7$; 6 inches.
12. $1\frac{2}{3}$ inch from top; $6\frac{2}{3}$ inches.
13. (i) $y = k/x + 3$; (ii) $y = k/(x + 4{\cdot}7)$;
(iii) $y = k/(x - 5) - 2$.
14. (i) $k = 28$; (ii) $k = 3{\cdot}6$; (iii) $k = 72$.
15. $y = 60/(x - 6)$.
16. (i) $(y + 5)(x + 2) = 30$, i.e. $y + 5$ is inversely prop. to $x + 2$;
(ii) $(y - 3)(x + 7) = 100$ or $y = 100/(x + 7) + 3$;
(iii) $(3y + 2)(2x + 3) = 138$ or $y = 46/(2x + 3) - \frac{2}{3}$;
(iv) $(y - 20)(5x - 2) = 40$ or $y = 40/(5x - 2) + 20$.
17. (i) 1; (iii) $1\frac{1}{3}$; (iv) $1{\cdot}2$.
18. (2, 4). 19. (2, 4).

EXERCISE XXIV.

See ch. XI., § 1; ch. XVI., B, C.

A.

1. 3·75 tns.; 3·5 inches.
2. From graph when $h = 20$, $d = 5·45$ ∴ when $h = 20 \times 100$ $d = 54·5$. Similarly, when $d = 5$ $h = 16·8$ ∴ when $d = 50$ $h = 1680$ feet.
3. Charge is directly proportional to cube of half-thickness.
4. (i) 2·2 lb.; (ii) 2·7 feet.
5. (i) $y = 0·1x^2$; (ii) $y = 1·9\sqrt{x+4}$; (iii) $y - 3 = 5\sqrt[3]{x+7}$. Assume (i) $y = kx^2$, (ii) $y = k\sqrt{x+4}$, etc., and substitute for x and y.
6. $y = 2\sqrt{(x-1)}$. 7. $y = 2(x+3)^2$.
8. (i) $x = 5, y = 15$; (ii) $x = 8, y = 1·6$; (iii) $x = 8, y = 6\sqrt{2}$; (iv) $x = 2, y = 50$.
9. The curve of $y = 2\sqrt{(x+10)}$ must be obtained by plotting points.

B.

10. The loss of steam-pressure is inversely proportional to the square root of the diameter of the cylinder. (i) 4·3 lb.; (ii) 25 in.
11. $w = 12/d^2$.
13. (i) $y = 12/(x-3)^2$; (ii) $y - 4 = 2/\sqrt{(x-10)}$; (iii) $y = 24/x^3$; (iv) $y = 24/\sqrt[3]{x}$.
15. (i) The x-axis and the ordinate at $x = 3$. (ii) The line parallel to the x-axis where $y = 4$ and the ordinate where $x = 10$.
16. (i) $y = 48/(x+2)^2$; (ii) $y = 21/\sqrt{(2x-3)}$.
17. (i) $x = 4, y = 6$; (ii) $x = 10, y = 6$; (iii) $x = 26, y = \frac{1}{6}$.

EXERCISE XXV.

See ch. XI., § 1; ch. XVI., D.

2. (i) $z = kx^2/y$; (ii) $z = k\sqrt{x}/y^2$; (iii) $z = k^3\sqrt{x/y}$; (iv) $z = k/x\sqrt[3]{y}$.
3. (i) $k = 1$; (ii) $k = 48$; (iii) $k = 2·1$; (iv) $k = 0·7$.
4. $d = 0·0175\, rl \cos\lambda = 361$ miles.
5. $M = 0·18d^3 \tan\alpha = 6158$.
6. $d^3 = (93)^3T^2$; (i) $d = 93\sqrt[3]{144} = 93\sqrt[3]{5^3 + 19} = 93 \times 5·26 =$ 489 millions of miles.
   (ii) $T^2 = (2780/93)^3 = (30)^3$ approx. $= 270 \times 100$.
   ∴ $T = \sqrt{270} = 10\sqrt{(16)^2 + 14} = 10(16 + \frac{14}{32}) = 165$ years.
7. $y = ax + b/x^2$.
8. $a = \frac{1}{2}, b = 8$.
9. $y = a\sin\alpha/(x-1) - bx\cos\alpha/(x-2)^2$.
10. $a = 20, b = 1$.
11. $\tan\delta = 18nCA/d^3 = 0·02$ ∴ $\delta = 1°$.
12. See ch. XVI., D., § 2.

EXERCISE XXVI.

See ch. XI., § 4; and (for D) ch. IV., § 7.

A.

1. (i) High; (ii) $s = \sqrt{32\cdot 2d}$; (iii) 11·3 ft./sec.; (iv) 34 ft./sec.
2. $d = s^2/32\cdot 2 - 3h$; 26·2 feet.
3. (i) $(a + 2)(b - 3)$; (ii) $(p + 3q)(x - 2y)$;
(iii) $(p + q)(p + q) - 1$; (iv) $(5 - p)(1 + p)$.
4. (i) 18·6; (ii) 4; (iii) $\frac{4}{11}$; (iv) 8.
6. 11·3. 7. $ab(p + q)$.
8. (i) $y = 2x - 3$; (ii) $y = 3 - 2x$; $(1\frac{1}{2}, 0)$ is common.
9. 32° 20′ E. of S.
10. 105 miles.

B.

1. (i) $P = \dfrac{9KC}{d}\left(1 + \dfrac{25}{d^2}\right)$; (ii) 100/25 = 4 lb.;
(iii) greater, for $k\alpha/90$ is greater for gunpowder;
(iv) 36,000 lb.; yes; (v) 61,250 lb.
2. (i) $C_1 = \frac{1}{2}(R_1/R + 1)(C_2 + C_3)$;
(ii) $C_2 = 2C_1R/(R_1 + R) - C_3$;
(iii) Interchange C_2 and C_3.
3. $a^2 + b^2 = (4n^2 + 1)^2$. Triads are 4, 3, 5; 8, 15, 17; 12, 35, 37; 16, 63, 65.
4. (i) $(2a - 3b + 4c)(2a - 3b - 4c)$;
(ii) $(3p - 2q - 5)(3p - 2q + 5)$;
(iii) $(p - q)(p + q\,a - b)/(pa - b)(qa - b)$;
(iv) $3(a - 1)/(a - 2)(a - 4)$.
5. (i) $\frac{32}{27}$; (ii) 6; (iii) $2\frac{1}{2}$; (iv) $\frac{1}{2}$.
6. 10 miles/hr.
7. (3, 4.)
8. $(p - q)x\{1 + (p + q)x\}$.
9. $y = 10 - 8 \sin x°$; (i) 10; (ii) 2; (iii) 3·368.
10. (i) 33°; (ii) 31°; (iii) 45°.

C.

1. (i) 29·4 gals./min.; (ii) 376·3 gals./min.
2. (i) $l = 864\cdot 4\ Hd^5/G^2$; (ii) $H = 0\cdot 001157\ G^2l/d^5$;
(iii) $d = 0\cdot 2586 \sqrt[5]{(G^2l/H)}$.
3. 3, 4, 5; 7, 24, 25; 21, 220, 221.
4. $6n^2(3n - 2)$. Expression vanishes for $n = 0$ and $n = \frac{2}{3}$.
5. (i) $\frac{1}{7}$; (ii) $a - b$; (iii) 7 and 2. (The value 2 does not satisfy the given condition as it stands. It actually satisfies $(n + 1)/(3 - n) - (n + 2)/(4 - n) = 1$.)
6. Tea, 2s.; coffee, 1s. 6d.
7. $w = kM/d^2$; $k = 16$; $w = 0\cdot 133$.
8. 218,000 miles.
9. 5/12 = tan 22½° approx.; 5/13 = tan 21° approx.
10. (i) 72⅓°; (ii) 59°; (iii) 10 feet; (iv) 6 feet; (v) 25 feet.

D.

1. Median, 59·5; mean (A.M.), 60. 2. 1·5. 3. 1·57. Former is less.
4. (i) (a) 7, (b) 10, (c) 2·6, (d) 3·2;
(ii) (a) 16, (b) 15, (c) 3·22, (d) 3·33;
(iii) (a) 13, (b) 13, (c), 13·08, (d) 13·08.
5. Median is each case = 10·5, but B's mean deviation (11·0) is less than A's (11·7); B is, therefore a better shot.
6. 13·25. Mean deviation, 3·67.
7. 15,0; 3·83.
8. Median, 13 years 8 months; mni. deviation, 7 months.
9. Median (i) 6·3; (ii) 17·5. Quartiles: (i) 4·7, 7·2;
(ii) 16, 19·65. Quartile deviation: (i) 1·25; (ii) 1·825.
10. The latter, because less dispersed.
11. The former, because *ratio* of dispersion to median is less.
12. The latter, because $2{\cdot}1/822 < 2{\cdot}3/824$.
13. Mi. 3·133; Q.D. 0·0595. 14. Mi. 2·5; Q.D. 0·082.
15. Mi. 1·985; Q.D. 0·185.
17. Mi. 0·5; Q.D. 0·045. 19. Mi. 1·505; Q.D. 0·099.

E.

1. (i) 0·449; (ii) 0·2245; (iii) 1·447; (iv) 0·598.
2. (i) 23·319; (ii) 5·481; (iii) 24·179.
3. (i) 5·828; (ii) 0·072; (iii) 9·898; (iv) 32·584; (v) 75·944.
4. (i) $7 + 5\sqrt{2} = 14{\cdot}070$; (ii) $26 - 15\sqrt{3} = 0{\cdot}02$;
(iii) $48\sqrt{3} - 38\sqrt{2} = 29{\cdot}404$.
5. (i) 3; (ii) 4; (iii) 1; (iv) 19; (v) 31; (vi) 13.
6. (i) $a - b$; (ii) $4a - 9b$; (iii) $ab - 1$; (iv) $pq - r$;
(v) $p/2 - q/3$; (vi) $1 - ab/5$; (vii) $a^2 - b$; (viii) $a^2 - 9b$.
7. (i) $\sqrt{3} - \sqrt{2}$; (ii) $\sqrt{3} + \sqrt{2}$; (iii) $(3 + \sqrt{5})/4$;
(iv) $(\sqrt{7} + 2)/3$; (v) $(2\sqrt{3} - 1)/11$; (vi) $(3\sqrt{5} + 1)/44$;
(vii) $2(6\sqrt{11} + \sqrt{5})/391$; (viii) $14 - 2\sqrt{35}$;
(ix) $(10\sqrt{33} + 24)/227$.
8. (i) $\sqrt{2}(a + \sqrt{2})/a(a^2 - 2)$; (ii) $4(3p + 4\sqrt{3})/p(9p^2 - 48)$;
(iii) $(5a - b\sqrt{5})b/5a(5a^2 - b^2)$; (iv) $p\sqrt{2}(3 + p\sqrt{3})/3(3 - p)$.
9. (i) 2·995; (ii) 12·109; (iii) 0·162; (iv) 1·118; (v) 0·528.
10. 12·36 cms.

F.

1. (i) 4·8 feet; (ii) 2 feet nearly.
2. $v = \sqrt{[58{\cdot}6\{R/(p^2 - 1) - 0{\cdot}05\}]}$.
3. The digits of the difference are $(a - c - 1)$, 9, $(10 - a + c)$. These reversed give the number $(10 - a + c)$, 9, $(a - c - 1)$. The sum of the two numbers is always 1089.
4. (i) $(2n + 1)/(2n^2 + 2n + 1)$;
(ii) $2n(n + 1)/(2n^2 + 2n + 1)$. [See C, No. 3.]
5. $(a^2 + ab + b^2)(a^2 - ab + b^2)$.
6. (i) $10/(2a - 3)$; (ii) $\frac{1}{2}$; (iii) 4.
7. 4/7. 8. (i) $n > 2\frac{1}{3}$; (ii) $n > \frac{1}{4}$; (iii) $n < 1/2\sqrt{3}$.
9. 8; £1 10. 10. 36·55 yards = 109·65 feet; 188·4 yards.

G.

1. (i) 1162 cu. ft./hr. ; (ii) 1741·5 cu. ft./hr.
2. (i) $H = Qgl/1000d^5$; (ii) $d = \sqrt[5]{(Qgl/1000H)}$.
4. (i) $(4a^2 + 6ab + 9b^2)(4a^2 - 6ab + 9b^2)$;
   (ii) $\left(a + 1 + \frac{1}{a}\right)\left(a - 1 + \frac{1}{a}\right)$; (iii) $(p + q - 1)(p - q + 1)$.
5. (i) 26/121 ; (ii) $8\sqrt{3}/121$.
6. $2(a^2 + b^2) = 12$. 7. (i) 6 ; (ii) $\frac{1}{6}$; (iii) 13 ; (iv) 1 ; 1.
8. 700. 9. 110 ft./sec., 10 ft./sec.
10. (i) 50° ; (ii) 26·1 inches.

H.

1. (i) 1·44 feet ; (ii) 12 feet.
2. (i) $B = E/d - Gs/(G + s) - R$;
   (ii) $G = s(R - r - B)/(s - R + r + B)$.
4. (i) $(4n^2 - 1)/(4n^2 + 1)$; (ii) $4n/(4n^2 - 1)$. [See B, No. 3.]
5. 3/2. 6. (i) $3\frac{3}{8}$; (ii) c ; (iii) 10, 5 ; (iv) 8.
7. 132° and 228°. 8. $(x - 7)(y - 12) = 64$; $12\frac{1}{3}$ inches.
9. $3\frac{1}{7}$ cubic inches.
10. (i) $\sin a = \frac{3}{4}$, $a = 48° 40'$; (ii) 68° 21′ ; (iii) 53° 10′ ;
    (iv) 26° 30′, 59° 30′.

SECTION II.

EXERCISE XXVII.

See ch. XVII., § 1 ; ch. XVIII., A.

A.

1. 3rd floor. 2. (i) 27° ; (ii) 46·1°.
3. − 0 09, − 0·21, − 0·56, + 1·82, − 0·54, + 0·62
− 1·51, − 0·02, − 1·62, + 0·63, + 1·86, + 0·61.
4. £15 8s. 3d.
5. + 0·3, + 0·13½, + 0·12½, + 0·4, − 0·3, − 0·2½,
+ 0·9, + 0·14½, + 0·9, 0, − 0·4, 0,
+ 0·3½, + 0·6, 0, − 0·10, − 0·17, − 0·11,
+ 0·6, + 0·½4, − 0·5, − 0·15, − 0·16, − 0·5.

B.

7. (i) − 5 ; (ii) − 10·3 ; (iii) − 7·92 ; (iv) − £2 15s. ;
(v) + $4\frac{5}{16}$ lb. ; (vi) − 4 hours 24½ minutes ; (vii) + 68·25°.
9. (i) 20 ± 7 ; (ii) 80 ± 14 ; (iii) 59½ ± 21½ ;
(iv) 10·15 ± 2·45 ; (v) 1·24 ∓ 0·81.
10. (i) £107 19s. ± £28 13s. ; (ii) £44 ± £20 13s. ;
(iii) £5 4s. 10d. ± £3 10s. 2d. ; (iv) 3 cwt. 11 lb. ∓ 106 lb.
(v) 12 hours 57 minutes ± 1 hour 48 minutes.
11. (i) 44 ; (ii) 2·6 ; (iii) £171 12s. ; (iv) 7¼ cu. feet ;
(v) 65·4°.
12. (i) − 5 ; (ii) + 3 ; (iii) − 3 ; (iv) + 15·8 ; (v) − 20 ;
(vi) − 13·35 ; (vii) − 21·25.
13. (i) + 4 ; (ii) + 47 ; (iii) + 27 ; (iv) − 12·2 ; (v) − 26·3 ;
(vi) − 29 ; (vii) − 6·8.

C.

14.	34·9 ± 3·2	31·4 ± 1·5	64·6 ± 7·3
	39·1 ± 4·9	31·95 ± 1·85	65·8 ± 12·7
	31·8 ± 5·0	33·5 ± 2·5	65·8 ± 11·2
	26·2 ± 2·6	34·65 ± 4·35	62·8 ± 9·3
	24·8 ± 2·4	31·15 ± 7·65	64·2 ± 11·6
	27·4 ± 3·3	30·9 ± 7·3	63·8 ± 15·1
	28·7 ± 5·4	34·1 ± 7·1	62·1 ± 11·2

15. 58° correct average. 16. Correct average 47°.

17. 52°. 18. 3 hours $5\frac{5}{11}$ minutes.
19. (i) 100 ; (ii) 15·3 ; (iii) + 5 ; (iv) − 3.
20. 30·41° ; 32·52° ; 64·16° ; fourth week in January.
21. 3·83° ; 4·61° ; 11·2°. 23. 24 hours 51 minutes.
25. 24 hours 27 minutes.
27. 13 paces north of the starting-point.
28. (i) 59 yards N. and 20 yards W. of starting-point ;
(ii) 3 yards S. of starting-point.
29. 80 feet N., 260 feet E., 50 feet down (or 70 feet above ground).

EXERCISE XXVIII.

See ch. XVII., §§ 2, 3 ; ch. XVIII., B.

A.

1. Oral. 2. R = (+ 7) + (− 3) = + 4.
3. R = − 4, i.e. fourth floor. 4. R = + 47. 5. R = + 40.
6. R = + 17. 7. p = (+ 9) − (− 3) = + 12.
8. He rode back p = − 3.
9. The first component p = (+ 14) − (− 2) − (+ 6) = + 10 ; ∴ turning is 8 miles out.
10. (i) R = + 2 ; (ii) R = + 7 ; (iii) R = − 7 ; (iv) d = + 4 ; (v) d = + 9 ; (vi) d = − 15 ; (vii) k = − 1 ; (vii) m = + 8.

B.

11. S = + 150 + 7·5t ; (i) £180 ; (ii) £105.
12. A = + 243 − 12t ; (i) £411 ; (ii) £3.
13. T = + 1317 − 5t ; (i) £1377 ; (ii) £1277.
14. C = + 620 − 4t ; (i) 524 tons ; (ii) 716 tons.
15. See ch. XVII., § 2 ; (i) 160 tons ; (ii) 30 hours hence ; (iii) 45 hours ago ; (iv) 48 hours ; (v) 6 days.
16. B = + 33 − 7t ; (i) balance of £61 ; (ii) deficit of £9.
17. B = − 8 + 4t ; (i) deficit of £28 ; (ii) balance of £12.
18. P = + 144 + 16t ; (i) − 240, i.e. £240 of capital not repaid ; (ii) + 216, i.e. total profit of £216.
19. P = − 120 + 24t ; (i) − 408, i.e. £408 of capital not repaid ; (ii) + 744, i.e. total profit of £744.
20. (i) 9 months ago, 5 months hence ;
(ii) 2 years 9 months hence.
21. £432, £840. 22. Column graphs needed (ch. IV., § 4).
23. h = + 300 + 0·06d ; (i) 345 feet ; (ii) 273 feet.
24. D = + 1000 + $\frac{1}{7}$d ; (i) 930 feet ; (ii) 1200 feet.
25. T = + 110 + 0·02d ; (i) 130° ; (ii) 94°.
26. T = + 50 + 0·02d.
27. d = 50(T − 50) ; (i) 5500 feet ; (ii) 2050 feet.
29. B = + 29·8 − 0·0011d ; (i) 30·625 inches ;
(ii) 28·623 inches.
30. B = 30·625 − 0·0011h.
31. h = 909(30·625 − B) ; (i) 1386 feet ; (ii) 295 feet.

C.

32. (i) − 23 ; (ii) + 11 ; (iii) − 48 ; (iv) + 57 ; (v) − 13 ; (vi) − 18.
33. (i) $A = p + 2q - 3r = +4$; (ii) $C = 10p - 6q = +136$;
(iii) $n = 5p + 2q + 2r = +34$;
(iv) $E = -\frac{1}{12}p + \frac{1}{6}q = -1\frac{5}{6}$;
(v) $B = 3p + \frac{2}{5}q - 4r = +40{\cdot}4$;
(vi) $M = 10{\cdot}3p - 6{\cdot}8q + 0{\cdot}2r = +143{\cdot}6$.
34. (i) $16n$; (ii) $9a - 12{\cdot}3b$; (iii) $-1{\cdot}6p - 3{\cdot}9b + 3{\cdot}2$;
(iv) $-\frac{2}{3}a + \frac{1}{2}b + 5\frac{3}{5}c$; (v) $-6a - 21b$.
35. (i) $3a - 6b + 6c$; (ii) $-3a + 6b - 6c$; (iii) $2a - 4b + 4c$;
(iv) $-1{\cdot}6p + 1{\cdot}6q - 101$; (v) $2q - r - 1$.

EXERCISE XXIX.

See ch. XVII., §§ 2, 4 ; ch. XVIII., C.

1. $d = +32 + vt$; (i) $+119\frac{1}{2}$; (ii) $-55\frac{1}{2}$; (iii) -37 ; (iv) $+65$; (v) °. [+ means north and − south of Doncaster.]
2. $d = d_0 + vt$; (i) $d = +250 + (-12)(+72) = -614$, i.e. 614 miles west of St. Paul ;
(ii) $d = -120 + (-15)(+15) = -345$;
(iii) $d = -120 + (-15)(+10) = +30$, i.e. 30 miles east of St. Paul.
3. $d = d_0 + ht$; $d = +13 + (-\frac{1}{4})(-12) = 16$ feet.
4. $d = d_0 + 4\,im$; $d = +400 + 4(-2{\cdot}4)(+4) = +361{\cdot}6$ fms.
5. $h = h_0 + 5280m \tan \alpha°$;
(i) $h = +2500 + 5280(-2\frac{1}{2})(-0{\cdot}123) = +876$ feet ;
(ii) + 4448 feet.
6. $d = p + q \cos \alpha°$; $d = -1{\cdot}5 + (+2{\cdot}25)(+0{\cdot}8) = -0{\cdot}5$, i.e. half a mile south.
7. $B = B_0 + pt$; (i) $B = -130 + (+20)t$; (ii) overdraft of £30 ; (iii) overdraft of £250.
8. $v = u + at$; $v = +27 + (-3)t$; (i) 12 mls./hr.; (ii) 39 mls./hr. ; (iii) at rest ; (iv) 6 mls./hr. in opposite direction.
9. $d = v_1t_1 - v_2t_2$; $d = (+24)(-1\frac{1}{2}) - (-20)(+2\frac{1}{2}) = 14$ miles.
10. $t = (v - u)/a$; (i) $t = \{(+8) - (+28)\}/(-4) = 5$ minutes ; (ii) $t = \{(-14) - (+28)\}/(-4) = 10\frac{1}{2}$ minutes.

B.

11. Oral. 12. $T_n = +10 + 3n$; + 160. 13. $T_n = +10 - 3n$; − 170.
14. $T_n = +10 + 3n$; (i) $n = -100$, $T_n = -290$; (ii) − 590.
15. $T_n = +8 - 5n$; (i) − 92 ; (ii) + 68.
16. $T_n = -4 + 4n$; (i) − 4004 ; (ii) + 3996.
17. (i) − 19, − 16, − 13, − 10 ; (ii) = 4, − 1, + 2, + 5 ; (iii) $T_n = -7 + 3n$; (iv) − 127 ; (v) + 38.

18. (i) $+24, 0, -24, -48$; (ii) $-96, -130, -154, -178$; $+11{,}928$.
19. **Oral.** (i) $-317{\cdot}2, +166{\cdot}7$; (ii) $-623{\cdot}5, -50{\cdot}125$.
20. (i) $T = -3 + 18n$; (ii) $T_n = -8{\cdot}6n$;
(iii) $T_n^n = +1\frac{1}{2} - 3\frac{3}{4}n$; (iv) $T_n = 0{\cdot}9 + 0{\cdot}07n$.
21. $T_n = T_o + dn$.
22. $d = 6{\cdot}7, T_o = -117, T_n = -117 + 6{\cdot}7n$.
23. $T_n = -10{\cdot}8 + 1{\cdot}25n$. 25. $T_n = -22{\cdot}3 - 2{\cdot}7n$.
26. $T_n = -53 - 3n$.
27. $d = (u - v)/(p - q)$;
$T_o = u - p(u - v)/(p - q) = v - q(u - v)/(p - q)$
$= (pv - qu)/(p - q)$; $T_n = \{(pv - qu) + n(u - v)\}/(p - q)$.

C.

28. $d = 4$; 7, 11, . . . 27. 29. $-1, -5, -9, -13, 17$.
30. $d = (l - a)/(n - 1)$. 31. $d = (l - a)/(n + 1)$.
32. **Oral.** 33. $-47{\cdot}3, +10{\cdot}3, -40{\cdot}9$.
34. 8, 53. 35. 15.
36. 7th, no, 9th, 13th, no.
37. Between 32nd and 33rd.
38. Yes, 3d ; diff. = pd. 39. Diff. = $d/(p + 1)$; $n + (n - 1)p$.
40. **Oral.**

EXERCISE XXX.

See ch. XVII., §§ 5, 6 ; ch. XIX., A and B.

A.

2. (i) 500,000,500,000 ; (ii) $n(n + 1)/2$; (iii) 10000 ;
(iv) n^2 ; (v) $n(n + 1)$.
3. (i) $+816$; (ii) $+120$; (iii) -127 ; (iv) $178\frac{1}{2}$.
4. (i) -12300 ; (ii) -903 ; (iii) $+20{\cdot}4$; (iv) $+192$.
5. (i) $+1326$; (ii) $+151\frac{1}{2}$; (iii) -990 ; (iv) $-112{\cdot}2$.
7. £2550. 8. 496 yards. 9. £114 8s.

B.

10. (i) £150 ; (ii) £2·27 . . . ; (iii) £0·54 . . . 11. 20 gallons.
12. 20 gallons. 13. (i) 20 gallons ; 1·6 gal./min.
15. 6·72 gals./min. 16. 8·712 gallons. 18. (i) 3 feet 9 inches ;
(ii) 15 feet ; (iii) $v = 0{\cdot}3t$; (iv) $s = 0{\cdot}15t^2$.
19. (i) See ch. XVII., § 6 ; (ii) $v = 5t$; (iii) $s = \frac{1}{24}t^2$;
(iv) 40 mls./hr. ; (v) $2\frac{2}{3}$ mls.
20. (i) $s = \frac{1}{2}at^2$; (ii) $s = 30at^2$. 21. (i) $v = 15 + 3t$;
(ii) $v = \frac{1}{4} + \frac{1}{20}t$; (iii) $s = \frac{1}{4}t + \frac{1}{40}t^2$.
22. (i) $s = ut + \frac{1}{2}at^2$; (ii) $s = (ut + \frac{1}{2}at^2)/60$.
24. (i) $v = 40 - 8t$; (ii) $s = (40 - 4t^2)/60 = \frac{2}{3} - \frac{1}{15}t^2$.
26. (i) After 6 seconds ; (ii) 54 feet ; (iii) 18 ft./sec. downwards ;
(iv) at starting-point ; (v) -42 ft./sec. ;
(vi) 240 feet below starting-point.

28. (i) 50 seconds; (ii) 40,000 feet; (iii) 960 ft./sec;
(iv) 25,600 feet high; (v) 320 ft./sec. downwards;
(vi) 38,400 feet high.
29. (i) $s = +7\cdot2t + 3\cdot2t^2$; (ii) $s = -10t + 4\cdot3t^2$;
(iii) $s = -8\cdot5t - 1\cdot8t^2$.
30. (i) Speed in cms./sec.: A, $-56\cdot8$; B, -96; C, $+27\cdot5$.
(ii) Dist. from origin in cms.: A, $+248$; B, $+530$; C, -95.
(iii) Speed in cms./sec.: A, $+0\cdot8$; B, $-18\cdot6$; C, $-4\cdot9$.
(iv) Dist. from origin in cms.: A, -4; B, $+14\cdot3$; C, $+6\cdot7$.
(v) A $1\frac{1}{8}$ seconds ago and then moved + ly;
B $1\frac{7}{40}$ seconds hence and will then move + ly;
C $2\frac{13}{36}$ seconds ago and then moved − ly.

C.

31. 10·4 sq. cms. (approx.). 32. 13·8 sq. cms. (approx.).
33. About 1100 cu. inches. 34. As No. 33. Areas of *all* equidistant sections assumed in A.P.
35. About 2450 cu. feet. 36. About 1300 c.cs.

EXERCISE XXXI.

[See ch. XVII., § 7; ch. XX., A and B.

B.

6. Expr. = (i) $6a^2 + 5ab - 6b^2$; (ii) $8p^2 - 18pq + 9q^2$;
(iii) $6x^2 - 23xy + 20y^2$;
(iv) $abx^2 + (a^2 + b^2)xy + aby^2$;
(v) $abx^2 - (a^2 + b^2)xy + aby^2$;
(vi) $abx^2 + (a^2 - b^2)xy - aby^2$;
(vii) $\frac{3}{8}x^2 - x + 1$; (viii) $x^2 - (pb/a + qa/b)x + pq$;
(ix) $6/a^2 + 5/ab - 6/b^2$;
(x) $ab/x^2 - (a^2 - b^2)/xg - ab/y^2$.
12. $n^3 + 1$. 13. $n^3 - 1$. 14. $n^4 + n^2 + 1$.
17. $a^3 + b^3 + c^3 - 3abc$.

C.

19. (i) 3756; (ii) 29631; (iii) 305206; (iv) 1020201; (v) 715; (vi) 90909.
20. $3t^3 + 8t^2 + 13t + 6$. 21. $8t^3 - 48t^2 + 76t - 24 = 3936$.
22. (i) $6t^6 - 13t^5 + 18t^4 - 17t^3 + 12t^2 - 7t + 2$;
(ii) $3t^4 - 8t^3 + 14t^2 - 8t + 3$; (iii) $t^5 - t^3 - t^2 + 1$;
(iv) $32t^5 + 1$; (v) $t^5 - 243$;
(vi) $6t^6 - 9t^5 - 4t^4 + 3t^3 + 9t^2 - 2t - 3$.
23. (i) $a^6 + b^6$; (ii) $a^6 - a^4b^4 + 2a^3b^3 + b^6$;
(iii) $x^3 + 3xy + y^3 - 1$; (iv) $1/a^4 - 1$;
(v) $16p^4/81q^8 + 4p^2/9q^4 + 1$.
24. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
25. (i) $a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$;
(ii) $a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$;

(iii) $4a^2 + 9b^2 + c^2 + 12ab + 6bc + 4ca$;
(iv) $4a^2 + 9b^2 + c^2 + 12ab - 6bc - 4ca$:
(v) $p^2 + 4q^2 + 9 - 4pq - 12q + 6p$;
(vi) $9p^2 + 4q^2 + 1 - 12pq - 4q + 6p$;
(vii) $a^2/b^2 + 4 + b^2/a^2 + 4a/b + 4b/a + 2$;
(viii) $a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2$;
(ix) $p^6 + q^6 + r^6 + 2p^3q^3 + 2q^3r^3 + 2r^3p^3$;
(x) $1/p^2 + 4/q^2 + q/r^2 - 4/pq - 12/qr + 6/rp$.

D.

26. 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1.
27. $a^7 \pm 7a^6b + 21a^5b^2 \pm 35a^4b^3 + 35a^3b^4 \pm 21a^2b^5 + 7ab^6 \pm b$?
28. (i) $1 + 3a + 15a^2/4 + 5a^3/4 + 15a^4/16 + 3a^5/16 + a^6/64$;
(ii) $32p^5 - 80p^4q/r + 80p^3q^2/r^2 - 40p^2q^3/r^3 + 10pq^4/r^4 - q^5/r^5$.
30. (i) $+ 4536a^4$; (ii) $+ 20p^4$; (iii) $- 2268a^6b^3$; (iv) $- 15p^6q^3r^3$;
(v) $+ 2835/8$; (vi) $- 280a^4b^3p$, $+ 112a^3b^4/p$.

EXERCISE XXXII.

See ch. XVII., § 8; ch. XXI., A.

A.

1. York, acreage $3 \cdot 7 \times 10^5$, pop. $39 \cdot 7 \times 10^5$; etc.
2. (i) $8 \cdot 3 \times 10^7$ acr., $5 \cdot 9 \times 10^6$ pop.;
(ii) $0 \cdot 7 \times 10^6$ acr., $4 \cdot 8 \times 10^6$ pop.;
(iii) average acr. $2 \cdot 1 \times 10^7$ and $0 \cdot 175 \times 10^6$ or 21×10^5 and $1 \cdot 75 \times 10^5$; (iv) pop. $1 \cdot 5 \times 10^7$ and $1 \cdot 2 \times 10^6$.
3. **Oral.** 4. **Oral.** $2 \cdot 3827 \times 10^4$, etc.

B.

5. $1 \cdot 47 \times 10^{13}$ miles. 6. $6 \cdot 77 \times 10^{26}$ lb. 7. $9 \cdot 2 \times 10^{16}$ cu. inches.
8. (i) $9 \cdot 21 \times 10^7$; (ii) $7 \cdot 21 \times 10^{13}$; (iii) $7 \cdot 96 \times 10^{10}$;
(iv) $5 \cdot 06 \times 10^{11}$; (v) $6 \cdot 72 \times 10^{10}$.

C.

9. $8 \cdot 33$ mins. 10. 3×10^{17} tons. 11. $3 \cdot 6 \times 10^2$ cu. miles.
12. (i) $2 \cdot 03 \times 10$; (ii) $6 \cdot 92 \times 10^9$; (iii) 2×10^2.

D.

13. (i) a^2b^3c; (ii) $a^2/b^2p^3q^5$; (iii) 1; (iv) $8p^3q^3r^3/125$; (v) a^6;
(vi) a^6; (vii) p^{126}; (viii) p^{126}.
14. (i) a; (ii) a^4; (iii) a^2/b; (iv) px/q^3; (v) abc; (vi) a;
(vii) p^2; (viii) p^2/q^7r^4; (ix) a^3b^5.
15. (i) $x^{2l}y^{2m}/z^{3n}$; (ii) x^{n+l}, y^{l+m}, z^{m+n};
(iii) a^{n+2l}, b^{l+2m}, c^{m+2n}; (iv) $a^{m(l-n)}$, $b^{n(m-l)}$, $c^{l(n-m)}$; (v) a^l.
16. (i) $3 \cdot 84 \times 10^9$; (ii) $1 \cdot 344$; (iii) $68 \cdot 6$.
17. (i) $2^6 \times 3^5 \times 5^4 \times 7^4$; (ii) $(2^4 \times 7 \times 13^2)/(3 \times 5 \times 17)$.
(iii) $(3^2 \times 11^4)/(2^5 \times 13^2 \times 19 \times 23)$.

EXERCISE XXXIII.

See ch. XVII., § 8; ch. XXI., B.

1. $2{\cdot}48 \times 10^{-2}$, $3{\cdot}72 \times 10^{-4}$, $6{\cdot}781 \times 10^{5}$, $6{\cdot}25 \times 10^{-1}$, $1{\cdot}875 \times 10^{-2}$, $3{\cdot}6 \times 10^{-4}$, $2{\cdot}3 \times 10^{-5}$, $2{\cdot}5 \times 10^{-8}$, 4×10^{-9}.
2. $6{\cdot}45 \times 10^{-1}$, $2{\cdot}31 \times 10^{-1}$, $1{\cdot}93 \times 10^{-1}$, 611×10^{-1}, or $6{\cdot}11 \times 10$.
3. $6{\cdot}67 \times 10^{-8}$ seconds. 4. $1{\cdot}18 \times 10^{-6}$ lb. 5. $4{\cdot}61 \times 10^{-5}$ lb.

B.

6. (i) 4×10^{-4}; (ii) $1{\cdot}74 \times 10^{2}$; (iii) $1{\cdot}11 \times 10^{2}$; (iv) $1{\cdot}6$.
7. (i) $2^{-1} \times 17^{-1}$; (ii) $2^{-5} \times 13^{-2} \times 23^{-1}$;
(iii) $2^{-4} \times 11 \times 13 \times 19^{-1}$.
8. (i) a^3c^2/b^4; (ii) $2b^5/3a^7$; (iii) $20r^5/pq^5$;
(iv) $\{bc(b^2 - c^2) + ca(c^2 - a^2) + ab(a^2 - b^2)\}/a^2b^2c^2$
$= \{a^3(b - c) + b^3(c - a) + c^3(a - b)\}/a^2b^2c^2$.
9. (i) $4p^2qr^{-5}/5$; (ii) x^{-3}; (iii) $2a^{-3}b^3$;
(iv) $x^{2l - 3n}$, $y^{m - 3l}\ z^{2n - 5m}$; (v) $x^{m(l - n)}$, $y^{n(m - l)}$, $z^{l(n - m)}$.

EXERCISE XXXIV.

See ch. XVII., § 9.

A.

1. (i) $5p + 11q$; (ii) $a + 3b$; (iii) $4m - 3n$; (iv) $a - 1/b$;
(v) $2a + 3/b$; (vi) $p + 4$; (vii) $p^2 - 13$; (viii) $ax^2 - 10$;
(ix) $x^2 - 8a$; (x) $p/3q + 9$; (xi) $a^2 + 2a + 1$;
(xii) $a^2 + ab + \frac{1}{4}b^2$; (xiii) $4/p^2 - 4q/3p + q^2/9$;
(xiv) $4a^2 - 6a + 9$; (xv) $p^2/16 - p/4 + 1$;
(xvi) $a^2p^4 + ap^2q^2 + q^4$; (xvii) $1 + 5/3r + 25/9r^2$;
(xviii) $a + b + c$; (xix) $p - q + r$;
(xx) $a^2 + b^2 + c^2 + ab + bc + ca$;
(xxi) $a^2 + b^2 + c^2 + ab - bc - ca)$;
(xxii) $x^2 + y^2 + z^2 - xy + yz - zx$.
2. (i) $(a + 5)(a + 4)$; (ii) $(x - 5)(x - 4)$; (iii) $(a - 8b)(a - 7b)$;
(iv) $(px^2 + 13)(px^2 + 10)$; (v) $(4x^2 - p)(3x^2 - p)$;
(vi) $(3a + 2b)(2a + 3b)$; (vii) $(3a - 4b)(2a - 3b)$;
(viii) $(4x - 5)(2x - 7)$; (ix) $(2x + 7)^2$;
(x) $(4x - 5/y)^2$; (xi) $(a + 4)(a - 2)$;
(xii) $(a - 4)(a + 2)$; (xiii) $(p + 7q)(p - 4q)$;
(xiv) $(ap - 7)(ap + 4)$; (xv) $(5a - 1)(4a + 1)$;
(xvi) $(5p + q)(4p - q)$; (xvii) $(3x - 2)(2x + 3)$;
(xviii) $(3a + 2b)(2a - 3b)$; (xix) $(7a/p^2 - 4)(3a/p^2 + 5)$;
(xx) $(6 - 7p^2)(2 + 3p^2)$; (xxi) $(13 + 2a/x^2)(3 - 4a/x^2)$;
(xxii) $(2a - 3)(4a^2 + 6a + 9)$; (xxiii) $(1 + x/4)(1 - x/4 + x^2/16)$;
(xxiv) $(3a + 4b)(9a^2 - 12ab + 16b^2)$;
(xxv) $(p - 2/p)(p^2 + 2 + 4/p^2)$;
(xxvi) $(1/3a^2 + 5/b^2)(1/9a^4 - 5/3a^2b^2 + 25/b^4)$;
(xxvii) $(x^2 + xy + y^2)(x^2 - xy + y^2)$;

(xxviii) $(a^2/4 + 3a/2 + 9)(a^2/4 - 3a/2 + 9)$;
(xxix) $(n^2 + n + 1)(n^2 - n + 1)(n^4 - n^2 + 1)$.
(xxx) $(p + 2)(2p - 1)(p^2 - 2p + 2)(4p^2 + 2p + 1)$.

B.

3. (i) $a + b + c$; (ii) $a - b - c$; (iii) $p + 3q + 2r$;
(iv) $2a + 3b - 5c$; (v) $a/2 - b/3 + c/5$; (vi) $\frac{1}{2}p^2 + \frac{2}{3}q^2 + \frac{3}{4}r^2$;
(vii) $ab - 2bc + 4ca$.
4. (i) $a^2 - a + 2$; (ii) $a^2 + a - 2$; (iii) $a^2 - 2a + 3$;
(iv) $2a^2 + 5a - \frac{1}{3}$; (v) $\frac{1}{3}a^2 - a + 3$.
5. (i) $p - 2 + 1/p$; (ii) $p - 2 - 1/p$; (iii) $p - 1 + 1/p$;
(iv) $a/p + 1 - p/a$; (v) $p^2 - p + \frac{1}{4}$; (vi) $2a/3b + 3/4 + 4b/5a$.

EXERCISE XXXV.

See ch. XVII., § 9; ch. XXII., A.

A.

1. (i) $a^2 - 3a + 7$; (ii) $3a^2 + 11ab - 2b^2$; (iii) $a^2 - 7ax^2 - 11x^4$;
(iv) $a^3 - 2a^2 + 3a - 4$; (v) $2 + 7p - p^2 + 8p^3$; (vi) $3a^3 - 7a + 4$;
(vii) $7 + x - 9x^3$; (viii) $a^3 + 2a^2 - 3a - 3$;
(ix) $2a^4 - 3a^3 - 7a^2 - 3a - 7$; (x) $1 + 3a + 9a^2 + 6a^3 + 18a^4$
2. (i) $a - 4$; (ii) $2p - 3q$; (iii) $n^2 - 2n + 1$;
(iv) $a^3 + 3a^2 - 2a + 3$; (v) $p^3 - 3p^2 + 2p - 1$;
(vi) $2a^3 - 3a + 5$; (vii) $2a^3 - 3a^2 + 7$;
(viii)
(ix) $2 - 5x^2 + 3x^4$; (x) $1 + x - 2x^2 - 5x^3 + x^4$.
3. (i) $a^3 - a^2 + a - 1$; (ii) $a^3 - a^2b + ab^2 - b^3$;
(iii) $a^4 - a^3 + a^2 - a + 1$; (iv) $p^5 - p^4q + p^3q^2 - p^2q^3 + pq^4 - q^5$;
(v) $1 + a + a^2 + \ldots + a^8$; (vi) $1 - a + a^2 - \ldots + a^{12}$.

B.

6. (i) $P = a - 6$, $Q = +19$; (ii) $P = 2a + 9$, $Q = +26$;
(iii) $P = a^2 - 4a + 6$, $Q = -7$;
(iv) $P = 2a^2 + 5a + 9$, $Q = +48$; (v) $P = a^2 + 4$, $Q = +24$;
(vi) $P = a^3 - 2a^2 + 8a - 33$, $Q = +133$.
10. $+19$. 12. (i) $+79/(a + 3)$; (ii) 0; (iii) $+1/(a + 1)$; (iv) 0;
(v) $+3/8(2a - 3)$.
13. (i) $P = a^2 + a - 1$, $Q = -3a + 4$;
(ii) $P = a^2 + a + 2$, $Q = -2a$;
(iii) $P = 6a^2 + 7ab + 60b^2$, $Q = 164ab^3 + 426b^4$;
(iv) $P = a^3 - a^2 + 2a - 3$; $Q = 5a - 4$.
14. (i) $(a^2 + a - 1) - (3a - 4)/(a^2 - 2a + 3)$, etc.
15. $P = a^2 + 2a + 3$, $Q = (7 - p)a$. 16. $p = 7$. 17. 4.

C.

18. (i) $1 - a + a^2 + a^3 + 2a^4/(1 + a)$;
(ii) $1 - a + a^2 - \ldots + a^6 - 2a^7/(1 + a)$;
(iii) $1 + a + a^2 + \ldots + a^5 + 2a^6/(1 - a)$.
(iv) $1 + a + a^2 + \ldots + a^8 + 2a^9/(1 - a)$.

19. 2/11000. 20. (i) 1/1458; (ii) 3/1093.
21. Error 1/128 is 2/513 of whole value.
25. (i) $1 - a + a^2 - \ldots - a^5 + a^6/(1 + a)$;
(ii) $1 - a^2 + a^4 - \ldots + a^{12} - a^{14}/(1 + a^2)$;
(iii) $1 + 2a + 4a^2 + \ldots + 32a^5 + 64a^6/(1 - 2a)$;
(iv) $1 - a^2/3 + a^4/9 - \ldots + a^8/81 - a^{10}/81(3 + a^2)$;
(v) $1 - a - a^2 - \ldots - a^6 - a^7/(1 - a)$;
(vi) $1 - 3a + 6a^2 - 12a^3 + 24a^4 - 48a^5/(1 + 2a)$;
(vii) $1 + a/2 - a^2/4 + a^3/8 - \ldots - a^6/64 + a^7/64(2 + a)$;
(viii) $1 - 5a/6 + 5a^2/18 - 5a^3/54 + 5a^4/54(3 + a)$.
26. (i) $2 + 6a + 18a^2 + 54a^3 + 162a^4/(1 - 3a)$;
(ii) $3 - 3a/2 + 3a^2/4 - 3a^3/8 + 3a^4/16 - 3a^5/16(2 + a)$;
(iii) $\frac{1}{2} + a^2/4 + a^4/8 + \ldots + a^{10}/64 + a^{12}/64(2 - a^2)$;
(iv) $\frac{1}{3} - 2a/9 + 4a^2/27 - 8a^3/81 + 16a^4/81(3 + 2a)$;
(v) $\frac{5}{2} - 5a/4 + 5a^2/8 - 5a^3/16 + 5a^4/32 - 5a^5/32(2 + a)$;
(vi) $\frac{1}{4} + 5a/16 + 15a^2/64 + 45a^3/256 + 135a^4/256(4 - 3a)$.
29. $1 + 3a + 2a^2 - a^3 - a^4(5 - a)/(1 - a + a^2)$.
30. $1 - 2x + 3x^2 - 4x^3 + 5x^4 - x^5(6 + 5x)/(1 + x)^2$.

EXERCISE XXXVI.

See ch. XVII., § 9; ch. XXII., B.

A.

1. (i) 20 minutes; (ii) 19 minutes. 2. (i) 100 feet; (ii) 90 feet.
3. (i) $33\frac{1}{3}$ inches; (ii) between the 6th and 7th semi-vibrations.
4. £4000; £12 13s. 8d.
5. £5727; £611 11s. nearly.

B.

6. (i) 3/2; 1·88 . . . , 1·77 . . . , 2·66 . . . ; 20·25, 30·375, 45·59; (ii) − 3/4; + 26·37 . . . , − 14·22 . . . , + 10·66 . . . ; + 3·375, − 2·53 . . . , + 1·89 . . . ; (iii) 1/3; $4\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{2}$; 1/162, 1/486, 1/1458;
(iv) $-q/p$; $-p^5/q^6, p^4/q^5, -p^3/q^4$; $-1/p, q/p^2, -q^2/p^3$,
(v) $1/a(a-b)$; $a^4(a-b)^3, a^3(a-b)^2, a^2(a-b)$; $1/a^2(a-b)^3$, $1/a^3(a-b)^4$, $1/a^4(a-b)^5$.
7. $6 \times (3/2)^{12} = 3^{13} \times 2^{-11}$; $2^{13} \times 3^{-11}$.
8. $-3^8 \times 2^{-13}$; $(-)^n 2^{2n-1} \times 3^{-n+1}$.
9. q^7p^{-8}; $(-)^n p^n q^{-n+3}$. 10. $a^{-n+2}(a-b)^{-n+1}$; $a^n(a-b)^{n-1}$.
11. (i) 2; (ii) 2/3; (iii) 144; (iv) 25; (v) $b/(a+b)$; (vi) $(a+b)^3/2b(a-b)^2$.
12. (i) $(1/2)^{n-1}$; (ii) $25 \times (0·3)^n$; (iii) $(-)^n a^n b^{-n+1}(a-b)^{-1}$.
13. $K =$ (i) $(1/2)^{n-2}$; (ii) $(5/6)^n$; (iii) $(a-b)^n(a+b)^{-n}$.
14. $S_n =$ (i) $2^n - 1$; (ii) $5000\{(1·02)^n - 1\}$;
(iii) $32\{(-)^n(5/4)^n - 1\}/9$; (iv) $8\{(9/8)^n - 1\}/81$;
(v) $q^2\{(p^2/q)^n - 1\}/(p^2 - q)$.

15. (i) $T_n = ar^{n-1}$; (ii) $S_n = a(r^n - 1)/(r - 1)$;
(iii) $S_n = a(1 - r^n)/(1 - r)$; (iv) $S = a/(1 - r)$.

C.

17. $P = A \times (1 \cdot 03)^{-n}$; $P \equiv$ the sum which would amount to A in n years.
18. $A = P(1 + i)^n$; $A = P(1 + i)^{-n}$.
19. £106 10s. 20. £91 12s.
22. (i) $a = Ai/\{(1 + i)^n - 1\}$; (ii) $a = Pi\{1 - (1 + i)^{-n}\}$.
23. £61 19s. 2d. 24. £44 19s. 5d.
25. £3063 6s. 2d. 26. £55 11s. 10d. 28. £200,000. 29. £1000.
30. £720; $3\frac{1}{3}$ per cent.

EXERCISE XXXVII.

See ch. XVII., § 10; ch. XXIII.

A.

2. (i) P moves to left occupying -1, -2, -4, etc.
(ii) P vibrates from right to left, left to right, etc., with decreasing swings, occupying $+20$, -10, $+5$, $-2 \cdot 5$, etc.
3. (i) P moves in towards O, occupying -1, $-\frac{1}{2}$, $-\frac{1}{4}$, etc.
(ii) P vibrates with increasing swings, occupying $+20$, -40, $+80$, etc.
4. P occupies $+3625$, -1235, $+385$, -155, $+25$, -35, -15, $-21\frac{2}{3}$, $-19\frac{4}{5}$. As n rises P vibrates about -20 with constantly decreasing swings. As n sinks the swings become large without limit.
7. Exp. $= +10 + \dfrac{5}{1 + 3x}$.
8. Exp. $= -10 + \dfrac{25}{1 - 3x}$. P starts from -10 when $x = -\infty$ and moves to right. It eventually reaches -10 again from the left.
9. Exp. $= +10 + \dfrac{5}{1 + 3x^2}$. P starts at $+10$ when $x = -\infty$ and moves towards the right reaching $+15$ when $x = 0$. It then returns to $+10$.
10. $\dfrac{1}{x} - 4x = +\infty$ if $x = -\infty$. As x rises P moves to the left crossing O when $x = -\frac{1}{2}$. Between $x = -\frac{1}{2}$ and $x = 0$, P traverses the whole negative scale. Directly x becomes positive P reappears at $+\infty$ and again moves down the scale reaching O when $x = +\frac{1}{2}$. From $x = +\frac{1}{2}$ to $x = +\infty$ P moves along the negative scale.

B.

11. $x = -13$.
12. -5. See No. 11.
13. (i) $-2{\cdot}44$; (ii) $+1$. (iii) $-2{\cdot}2$; (iv) -2; (v) $+21{\cdot}5$; (vi) $+21{\cdot}5$; (vii) -7; (viii) $+7$; (ix) $-4\frac{1}{3}$; (x) $8x^2 - 8 = 0$ $\therefore x = \pm 1$; (xi) $-3x^2 + 12 = 0$, $x = \pm 2$; (xii) -1.
14. $\{(P + Q)x - (2P + Q)\}/(x^2 - 3x + 2)$.
15. $2x - 5$.
16. (i) $P + Q = 0$, $2P + Q = -1$; $\therefore P = -1$, $Q = +1$; (ii) $P = -1$, $Q = +2$; (iii) $P = +1 = Q$.
17. (i) $-3/2$ and 0; (ii) $+\frac{5}{6}$, $+\frac{2}{3}$; (iii) $-9/20$, $-3/20$.
18. (i) $3/(x + 3) - 2/(x + 2)$; (ii) $2/(x - 2) + 3/(x + 2)$; (iii) $1/(2x - 3) - 4/(x + 2)$; (iv) $6/(3x - 2) - 5/(4x + 7)$.
19. (i) -2; (ii) $+19/30$; (iii) $-35/99$; (iv) $+13/369$.
20. $P = 2$, $Q = 3$.
21. (i) $2/(x + 3) - 5/(x + 3)^2$; (ii) $1/(2x - 3) - 6/(2x - 3)^2$;
(iii) $3/(3x + 4) - 5/(3x + 4)^2$;
(iv) $2/(x + 2) - 8/(x + 2)^2 + 10/(x + 2)^3$.
22. (i) $3 + 5/(2x - 1) - 12/(x + 2)$;
(ii) $3 + \dfrac{4}{3x - 5} - \dfrac{12}{x + 2}$;
(iii) $1 + 2/(2x - 3) + 3/(2x - 3)^2$;
(iv) $4 - 3/(x + 1) + 2/(x + 1)^2 - 1/(x + 1)^3$.
23. (i) $\dfrac{3}{(x - 10)(x - 7)}$; (ii) $\dfrac{3}{(x - 6)(x - 9)}$.
From $\dfrac{3}{(x - 10)(x - 7)} = \dfrac{3}{(x - 6)(x - 9)}$
we have $(x - 6)(x - 9) = (x - 10)(x - 7)$; whence $x = +8$.
$(x - 9)/(x - 10) - (x - 6)/(x - 7) = \{1 + 1/(x - 10)\} - \{1 + 1/(x - 7)\} = 1/(x - 10) - 1/(x - 7)$.
Similarly with the other side. Hence $x = +8$ as before.
24. (i) 2; (ii) 4; (iii) -1.

C.

25. $x > +7/3$.
28. (i) $+7/3 < x < +6$; (ii) $-22/7 < x < +25/4$;
(iii) $x > +7$ or < -2; (iv) $-8 < x < +4$;
(v) $(x - 3)(x - 2) < 0$, i.e. $+2 < x < +3$;
(vi) $(3x - 2)(2x + 7) > 0$, i.e. $x < -3\frac{1}{2}$ or $+\frac{2}{3}$.
29. $m = -3$, $n = -7$.
30. (i) $+5, -3$; (ii) $+1, -1$; (iii) $-14/55, +26/61$;
(iv) $-4, +4$; (v) $-2, -2$; (vi) $+9/7, -2/7$;
(vii) $-3, +3, -1$; (viii) $-3\frac{1}{2}, -\frac{1}{4}$;
(ix) $+3, -7$ or $-7, +3$; (x) $-5, -20$ or $-20, -5$.

EXERCISE XXXVIII.

See ch. XVII., § 11.

A.

1. ·0019.
2. (i) 1 ; (ii) ·7761 ; (iii) ·6251 ; (iv) ·625.
3. (i) − ·008041 ; (ii) − ·00009399.
4. (i) 1 ; (ii) 15 ; (iii) − 5.
5. (i) 20° ; (ii) 61° 25′.

B.

6. $\dfrac{3a}{a^2 - 1}$.

7. $\dfrac{x}{x - 1}$; $\left\{1 - \dfrac{x - 1}{(x + 1)^2 + x - 1}\right\}\dfrac{x + 3}{x + 1} = \dfrac{x + 1}{x}$.

8. (i) $-8ab\dfrac{a^2 + 4b^2}{(a^2 - 4b^2)^2}$; (ii) $\dfrac{2}{a}$; (iii) $\dfrac{6x^2}{x^{12} - 1}$; (iv) 3.

13. (i) $\dfrac{1}{(1 - x)^2} = 1 + 2x$; 1·002 ;

(ii) $\dfrac{1}{(1 - x)^2} = 1 + 2x + 3x^2$; 1·002003 ;

(iii) $\dfrac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3$; 1·002003004.

14. (i) $\dfrac{1}{(1 - x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4$;

(ii) $\dfrac{1}{(1 - x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4$.

15. $$\frac{1}{(1 + x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots (-1)^r(r + 1)x^r + \frac{(-1)^{r+1}rx^{r+1}}{1 + x} + \frac{(-1)^{r+1}x^{r+1}}{(1 + x)^2}$$

$$\frac{1}{(1 + x)^3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4$$

$$\frac{1}{(1 + x)^4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4.$$

16. (i) 1·003006010015 ; (ii) ·997005990015 ; (iii) − ·98399968.

C.

17. (i) − 3 ; (ii) 0 or $-\frac{1}{2}$; (iii) $-\frac{1}{5}$; (iv) − 1.
18. (i) $m = 21\frac{197}{738}$, $n = -29\frac{295}{492}$; (ii) $m = 4$, $n = 1$;
(iii) $m = \frac{13}{2}$, $n = -\frac{13}{6}$; (iv) $m = \frac{1}{2}$, $n = -1$;
(v) $m = +3$, $n = -7$.
19. (i) − 1, + 1, + 3 ; (ii) − 5, 11, 17 ; (iii) 5, 6, 7 ;
(iv) 4, − 1, 0 ; (v) a, b, c.

20. (i) $\dfrac{2}{1-n} - \dfrac{3}{1+n^2}$; (ii) $\dfrac{1}{n} + \dfrac{2}{n-2} + \dfrac{3}{n-3}$;
(iii) $\dfrac{1}{2(1+n)} - \dfrac{1}{2(1-n)} + \dfrac{1}{(1-n)^2}$.

EXERCISE XXXIX.

See ch. xxiv., § 2 ; ch. xxv., A.

A.

Note.—Tangents are given to two decimal places only.

1. (i) Through origin at 58° to x – axis ;
(ii) ditto, raised 4·7 units ; (iii) ditto, lowered 8·2 units ;
(iv) through origin at 166° to x – axis raised 14·3 units ;
(v) through origin at 102° to x – axis lowered 7·8 units.
2. (i) $y = 0{\cdot}6x + 4{\cdot}7$; (ii) $y = +22 - 2{\cdot}05x$;
(iii) $y = 0{\cdot}9x - 12{\cdot}3$; (iv) $y = -0{\cdot}4x - 11$.
3. (i) $y = 0{\cdot}7x - 2{\cdot}6$, inclined at 35°, lowered 2·3 ;
(ii) $y = 0{\cdot}7x + 2{\cdot}5$, ditto, raised 2·5 ;
(iii) $y = 0{\cdot}75x + 3$, inclined at 37°, raised 3 ;
(iv) $y = -\frac{4}{3}x + \frac{20}{3}$, inclined 127°, raised $6\frac{2}{3}$;
(v) $y = -1{\cdot}8x + 8$, inclined 119°, raised 8 ;
(vi) $y = \frac{5}{9}x + 3$, inclined 29°, raised 3.
6. (i) $7x + 6y - 14 = 0$; (ii) $x - 3y + 7 = 0$;
(iii) $2{\cdot}9x + y = 0$; (iv) $x - 7 = 0$; (v) $y + 2{\cdot}3 = 0$.
7. (i) $(0, +\frac{7}{3})$; (ii) $130\frac{1}{2}$° nearly, $18\frac{1}{2}$° nearly ; (iii) 112° nearly ;
(iv) parallel to y – axis ; (v) parallel to x – axis. 8. $\frac{3}{4}$.
9. $x = -6$, function $(y) = +8$; graphs all pass through $(-6, +8)$.
10. – 5, – 1.

B.

12. (i) $y = 3x + 12$ or $y = 3(x + 4)$, 4 to left ;
(ii) $y = 2{\cdot}4x - 7{\cdot}2$ or $y = 2{\cdot}4(x - 3)$, 3 to right ;
(iii) $y = -\frac{1}{4}x + 5$ or $y = -\frac{1}{4}(x - 20)$, 20 to right ;
(iv) $y = -3{\cdot}1x - 24{\cdot}8$ or $y = -3{\cdot}1(x - 8)$, 8 to right.
13. (i) $y = 4x + 5$; (ii) $y = 4(x + 3) - 7$ or $y = 4x + 5$;
(iii) $y = -\underline{1}{\cdot}4(x + 5) + 6{\cdot}2$ or $y = -1{\cdot}4x - 0{\cdot}8$;
(iv) $y = -\bar{1}{\cdot}4x - 0{\cdot}8$; (v) $y = \frac{3}{2}(x - 4) + 3$ or $y = \frac{3}{2}x - 3$;
(vi) $y = \frac{3}{2}x - 3$.
14. (*a*) (i) – 8, (ii) – 15, (iii) – 9, (iv) – 28 ;
(*b*) (i) + 4, (ii) – 5, (iii) + 15, (iv) – 14.
15. $y = 2x + 13$.
16. (i) $y = 1{\cdot}5x - 11{\cdot}5$; (ii) $y = -2{\cdot}3x - 19{\cdot}5$;
(iii) $y = 0{\cdot}87x - 7{\cdot}22$; (iv) $y = -1{\cdot}6x + 8{\cdot}3$;
(v) $4x + 3y + 7 = 0$; (vi) $5x - 2y - 12{\cdot}4 = 0$.
20. (i) and (iii) yes ; (ii) no.

EXERCISE XL.

See ch. XXIV., § 8 ; ch. XXV., B.

A.

3. $= -\sin a$. 4. $-\cos a$, $\sin a$. 5. Both $= -1/\tan a$.
7. $-\cot a$, $-\tan a$. 8. $-\cot a$, $-\tan a$. 9. 1106 yards.
10. 2930 yards. 11. 3·4 miles, 12°.

B.

13. 144°, 15°, 21°.
14. $104\frac{1}{2}$,° 29°, $46\frac{1}{2}$°. 15. 31° ; 197·6, 105·8 yards.
16. 5366 yards nearly. 18. 69·15. 21. 36° 50′, 53° 10, 90°.
23. 1162 square yards nearly.
24. $2000\sqrt{1023} + 3000\sqrt{209} = 107{,}340$ square feet nearly.

EXERCISE XLI.

See ch. XXIV., § 7 ; XXIV., § 8.

A.

1. AC = 2·348, BC = 2·244 ; CD = 3·068, AD = 3·31 ; DE = 2·455, CE = 2·315 ; EF = 2·56, DF = 2·16 ; FG = 2·00, EG = 2·197 ; GH = 3·662, FH = 3·589 ; GK = 3·499, HK = 3·31.
2. 867 feet. 3. 1432 feet. 4. 1436 feet ; 4 feet difference.
5. 312 feet ; 314 feet.
6. C from A, 140° ; D from C, 34° ; E from D, 166° ; F from E, 37° ; G from F, 161° ; H from G, 53° ; K from H, 173°.
7. Partial results : C, − 1·8, + 1·51 ; D, + 2·544, + 1·716 ; E, − 2·38, + 0·594 ; F, + 2·044, + 1·54 ; G, − 1·89, + 0·651 ; H, + 2·204, + 2·925 ; K, − 3·285, + 0·403.
Final results : DC, − 1·8, + 1·51 ; D, + 0·744, + 3·226 ; E, − 1·636, + 3·82 ; F, + 0·408, + 5·36 ; G, − 1·482, + 6·011 ; H, + 0·722, + 8·936 ; K, − 2·563, + 9·339.

B.

9. $\text{Sin}\,220° = -\sin 40°$, $\cos 220° = -\cos 40°$, $\tan 220° = +\tan 40°$.
10. $\text{Sin}\,310° = -\sin 40°$, $\cos 310° = +\cos 40°$, $\tan 310° = -\tan 40°$.
11. (i) $\sin a = \sin(180° - a)$, $\cos a = -\cos(180° - a)$, $\tan a = -\tan(180° - a)$;
(ii) $\sin a = -\sin(a - 180°)$, $\cos a = -\cos(a - 180°)$, $\tan a = \tan(a - 180°)$;
(iii) $\sin a = -\sin(360° - a)$, $\cos a = \cos(360° - a)$, $\tan a = -\tan(360° - a)$.
12. Co-ordinates : B, − 139·8, + 207·25 ; C, + 318·6, + 430·8 ; D, + 474·8, + 80 ; E, + 852 9, + 454·3 ; F, + 1031·7, − 213·1 ; G, + 626·1, − 329·4.
13. About 9 yards.

14. AB 47° W. of N., BC 75° E. of N., CD 34° W. of N., DE 57° E. of N., EF 5° W. of N., FG 58° W. of N.
15. A : 0, 0 ; B : − 264·7, + 246·9 ; C : + 111·1, + 347·6 ; D : − 151·7, + 737·3 ; E : + 191·4, + 960 ; F : + 172·9, + 1171·2 ; G : 0, + 1279·3.

C.

17. $\cos(-48^\circ) = +0{\cdot}669$; $\sin(-48^\circ) = -0{\cdot}743$.
18 Sines : (i) − 0·985 ; (ii) + 0·342 ; (iii) + 0·866 ;
Cosines : (i) − 0·174 ; (ii) − 0·94 ; (iii) + 0·5.
19. (i) − 0·424 ; (ii) + 1 ; (iii) − 2·747 ; (iv) + 0·364.

EXERCISE XLII.

See ch. XXIV., § 3 ; ch. XXVI., A, B.

A.

3. (i) 9 to right ; (ii) 8 up ;
(iii) 11 down, 13 to right ; (iv) 9 up, 16 to left.

5. (i) $y + 3 = \frac{12}{x}$ or $y = 12/x - 3$; (ii) $y = \frac{12}{x+3}$;

(iii) $y - 5 = \frac{-7}{x}$; (iv) $y = \frac{-5}{x + 1\frac{1}{3}}$; (v) $y + 3 = \frac{6}{x+4}$;

(vi) $y + 3 = \frac{15}{x+5}$; (vii) $y + 3{\cdot}5 = \frac{9{\cdot}25}{x + 1{\cdot}5}$;

(viii) $y - \frac{2}{3} = \frac{\frac{4}{3}}{x - 2}$.

6.

	(a)	(b)
	(i) 3 up	$y = \frac{12}{x}$;
	(ii) 3 to right	$y = \frac{12}{x}$;
	(iii) 5 down	$y = \frac{-7}{x}$;
	(iv) $1\frac{1}{3}$ to right	$y = \frac{-5}{x}$;
	(v) 3 up, 4 to right	$y = \frac{6}{x}$;
	(vi) 3 up, 5 to right	$y = \frac{15}{x}$;
	(vii) 3·5 up, 1·5 to right	$y = \frac{9{\cdot}25}{x}$;
	(viii) $\frac{2}{3}$ down, 2 to left.	$y = \frac{4}{3x}$.

7. (i) $\pm 2\sqrt{3}$, $-3 \pm 2\sqrt{3}$; (iv) $-1\frac{1}{3} \pm \sqrt{5}$, $\mp \sqrt{5}$;
(viii) $2 \pm \frac{2\sqrt{3}}{3}$, $\frac{2}{3}(1 \pm \sqrt{3})$

8. (i) $y = \frac{36}{x - 4}$; (ii) $y + 3 = \frac{-40}{x - 2}$; (iii) $y - 10 = \frac{20}{x + 10}$.

B.

10. (i) head down (0, 7); (ii) head up (0, 0); (iii) head up (0, 7);
(iv) head down (0, − 12); (v) head up (0, 20);
(vi) head down (14, 0); (vii) head up (14, 0);
(viii) head down (− 16, − 21); (ix) head up (13, − 21);
(x) head up (− 7·8, 15·5).

11. $y = k(x + 6)^2 + 4$; $y = -k(x + 9)^2 - 11$;
$y = -k(x - 14)^2 - 7$; $y = kx^2 - 17$.

12.

	(a)	(b)	(c)
(i)	lower	4	7;
(ii)	upper	7	− 5;
(iii)	upper	− 1·8	7·2;
(iv)	upper	− 9·3	0;
(v)	lower	0	− 2·3;
(vi)	lower	− 21·3	− 7·8.

13. (i) $y = (x - 3)^2 - 6$; (ii) $y = (x + 5)^2 - 27$;
(iii) $y = -(x - 6)^2 + 43$; (iv) $y = 3(x - 2)^2 - 7$;
(v) $y = -7(x - 2)^2 + 17$; (vi) $y = (x + \frac{5}{2})^2 - 7\frac{1}{4}$;
(vii) $y = -2(x - \frac{7}{2})^2 + 27\frac{1}{2}$; (viii) $y = 6(x - \frac{7}{6})^2 - \frac{49}{6}$;
(ix) $y = -4(x - \frac{13}{8})^2 + \frac{169}{16}$; (x) $y = 2·3(x + 2·5)^2 - 21·575$.

14.

	(a)	(b)	(c)
(i)	lower	− 6	3;
(ii)	lower	− 27	− 5;
(iii)	upper	43	6;
(iv)	lower	− 7	2;
(v)	upper	17	2;
(vi)	lower	$-7\frac{1}{4}$	$-\frac{5}{2}$;
(vii)	upper	27·5	+ 3·5;
(viii)	lower	$-8\frac{1}{6}$	$1\frac{1}{6}$;
(ix)	upper	$10\frac{9}{16}$	$1\frac{5}{8}$;
(x)	lower	21·575	− 2·5.

15. (i) $y = (x + 7)(x - 1)$;
(iii) $y = 4\left(x - 2 + \frac{5\sqrt{2}}{2}\right)\left(x - 2 - \frac{5\sqrt{2}}{2}\right)$;
(v) $y = 3(x + 6 + 3\sqrt{3})(x + 6 - 3\sqrt{3})$;
(vi) $y = -3(x - 6 + 3\sqrt{3})(x - 6 - 3\sqrt{3})$;
(vii) $y = (5x - 3)(x + 4)$;
(viii) $y = 2(x - 1·6 + 2\sqrt{1·59})(x - 1·6 - 2\sqrt{1·59})$.

17. (i) − 7, 1; (iii) $2 \pm \frac{5\sqrt{2}}{2}$; (v) $-6 \pm 3\sqrt{3}$;

(vi) $6 \pm 3\sqrt{3}$; (vii) $\frac{3}{5}$, -4; (viii) $1{\cdot}6 \pm 2\sqrt{1{\cdot}59}$.
18. $y = -\frac{1}{3}(x-4)^2 + 7$. 19. $y = \frac{1}{3}(x-4)^2 + 7$.
21. $-2x^2 - 24x$.
22. $y = \frac{1}{4}(x+10)^2 - 9$; -10, -9. 23. $2x^2 - 24x - 26$.

C.

24. $y = -\frac{1}{10}(x-12)^2 + 25{\cdot}5$.
25. $y = \frac{1}{20}(x-10)^2 - 5$.
26. $y = -\frac{1}{12}(x-10)^2 + 20$.
27. $y = -\frac{1}{25}(x-35)^2 + 36$; 35, 36.
31. $y + \frac{1}{5} = \frac{1}{10}(x-6)^2$; $-\frac{1}{5}$. 32. Not parabolic.

EXERCISE XLIII.

See ch. XXIV., § 3; ch. XXVI., C.

A.

1. (i) $x = 0$ or 4, 3 or 1, $3{\cdot}29$ or $0{\cdot}71$;
(ii) $x = 2$ or -8, $-4{\cdot}41$ or $-1{\cdot}59$, -3, -3;
(iii) $x = 3{\cdot}74$ or $-0{\cdot}39$, $0{\cdot}49$ or $3{\cdot}01$;
(iv) $x = -1{\cdot}32$ or $0{\cdot}32$, 0 or -1.
2. (i) $5{\cdot}5$ or $-2{\cdot}5$; (ii) imaginary;
(iii) $0{\cdot}33$ or $-0{\cdot}67$; (iv) imaginary.
3. (i) $+3$, $+1$; (ii) $+5$, -1;
(iii) -6, -6; (iv) $+8$, $+5$;
(v) $+8$, -5; (vi) $+12$, $+3$;
(vii) 0, $+7$; (viii) 0, $-5{\cdot}3$;
(ix) $+0{\cdot}5$, $+3$; (x) $0{\cdot}66$, $+1{\cdot}5$;
(xii) $+0{\cdot}25$, $-3{\cdot}5$; (xiii) $+0{\cdot}2$, $+0{\cdot}33$;
(xiv) $-1{\cdot}5$, -2; (xv) $-0{\cdot}33$, $+2{\cdot}5$;
(xvi) $+2{\cdot}5$, $+2{\cdot}5$.
4. (i) $x^2 - 2x - 15 = 0$; (ii) $x^2 + x - 56 = 0$;
(iii) $x^2 - 7x + 12 = 0$; (iv) $x^2 + 20x + 100 = 0$;
(v) $x^2 - 100 = 0$; (vi) $x^2 - 2{\cdot}7x - 13 = 0$;
(vii) $x^2 - 2{\cdot}4x + 0{\cdot}63 = 0$; (viii) $x^2 - 6x - 16 = 0$;
(ix) $x^2 - 4{\cdot}6x + 3{\cdot}85 = 0$; (x) $x^2 - 10x + 22 = 0$;
(xi) $x^2 + 2x - 1 = 0$; (xii) $x^2 - 2{\cdot}6x - 3{\cdot}71 = 0$;
(xiii) $x^2 - 2ax + a^2 - b^2 = 0$; (xiv) $x^2 - 2pa^2x + p^2a^4 - q^2b^4 = 0$;
(xv) $x^2 - 2\sqrt{2}x - 1 = 0$; (xvi) $x^2 - 2\sqrt{5}x - 12 = 0$.
7. (i) $-\frac{5}{3}$, $-\frac{1}{3}$; (ii) $3{\cdot}65$, $2{\cdot}35$;
(iii) $2{\cdot}625$, $-1{\cdot}56$; (iv) $-\frac{q}{p}$, $\frac{v}{p}$;
(v) $\frac{a-b}{a+b}$, $-\frac{ab}{a+b}$.
8. (i) $12x^2 - 5x - 3 = 0$; (ii) $35x^2 - 29x + 6 = 0$;
(iii) $\sqrt{15}x^2 + (\sqrt{3} - \sqrt{5})x - 1 = 0$;
(iv) $abx^2 + (a+b)x + 1 = 0$;
(v) $a^3b^3x^2 - (a^3 + b^3)x + a^2b^2 = 0$;
(vi) $(p^2 - q^2)x^2 - 2(p^2 + q^2)x^2 + p^2 - q^2 = 0$.

9. (i) $\sqrt{3} \pm 1$; (ii) $-\sqrt{5} \pm 4$;
(iii) $\sqrt{13} \pm 2\sqrt{2}$; (iv) $-\sqrt{15} \pm 2\sqrt{6}$;
(v) $\sqrt{a} \pm \sqrt{a - b}$; (vi) $-\sqrt{a + b} \pm \sqrt{2b}$;
(vii) $\frac{1}{2}\sqrt{7} \pm \frac{3}{2}$; (viii) $\frac{1}{2}\sqrt{14} \pm \frac{1}{2}\sqrt{2}$.
10. (i) $-4{\cdot}5$; (ii) $+7{\cdot}6$; (iii) $-7{\cdot}7$; (iv) $+4{\cdot}3$;
(v) $6\frac{1}{3}$; (vi) $-22\frac{3}{7}$.

B.

11. (i) crosses; (iv) crosses; (vi) touches x-axis.
13. (i) $b^2 \geq 4ac$; (ii) $b^2 \geq 4ac$;
(iii) $b^2 = 4ac$; (iv) $b^2 \geq 4ac$; (v) $b^2 = 4ac$.
14. (i) Equal roots; (ii) unequal roots;
(iii) no roots; (iv) unequal roots;
(v) equal roots; (vi) unequal roots.
15. (i) $\frac{11}{2} \pm \frac{1}{2}\sqrt{73}$; (ii) 12 or -1;
(iii) $-2 \pm \sqrt{11}$; (iv) no roots;
(v) $-2 \pm 2\sqrt{6}$; (vi) no roots;
(vii) $\frac{13}{2} \pm \frac{1}{2}\sqrt{193}$; (viii) $\sqrt{11} \pm \sqrt{101}$;
(ix) no roots; (x) $\dfrac{\sqrt{18} \pm \sqrt{18 + 4\sqrt{3}}}{2}$.
16. $x^2 - xm(\alpha + \beta) + m^2\alpha\beta = 0$.
17. (i) $x^2 - 8x - 48 = 0$; (ii) $x^2 - 13x - 714 = 0$;
(iii) $x^2 + x - 6$.
18. (i) $x^2 + 4x - 21 = 0$, $x = 7$ or -3;
(ii) $x^2 - 19x + 84 = 0$, $x = 12$ or 7;
(iii) $x^2 - 11x - 102 = 0$, $x = 17$ or 6;
(iv) $x^2 + 9x - 190 = 0$, $x = -19$ or 10.
19. (i) $x = \frac{b}{a}$ or -1; (iii) $x = \frac{1}{p}$ or $-\frac{1}{q}$;
(iv) $x = -\frac{b}{a}$ or $-\frac{a}{b}$; (vi) $x = a \pm b$;
(vii) $x = \dfrac{a}{a - b}$ or $\dfrac{a + 2b}{a - b}$; (viii) $x = \dfrac{a^2 + b^2}{b - a}$ or $\dfrac{ab}{b - a}$;
20. (i) $x = 1{\cdot}25$; (ii) none; (iii) none; (iv) $x = 4{\cdot}302$, $0{\cdot}697$;
(v) none; (vi) none; (vii) none; (viii) none.

EXERCISE XLIV.

See ch. XXIV., § 5.

A.

1. $x^2 + 2bx + 4ac = 0$.
2. (i) $x = \pm\frac{1}{\sqrt{2}}$; (ii) $x = 5{\cdot}303$ or $1{\cdot}698$;

(iii) not possible; (iv) $x = 1{\cdot}39$ or $-2{\cdot}89$;
(v) $x = \pm 2{\cdot}309$; (vi) $x = \pm 1{\cdot}414$;
(vii) $x = -1{\cdot}5$; (viii) not possible.

3. $4{\cdot}464$.

4. (i) $+\frac{1}{2}, -\frac{1}{3}$; (ii) $x = -\frac{12}{37}$ or -5;
(iii) $x = -0{\cdot}64$ or $0{\cdot}39$; (iv) no values.

B.

6. (i) $x^3 - 2x^2 - x + 2 = 0$; (ii) $x^3 - 6x^2 + 11x - 6 = 0$;
(iii) $x^3 - 4x = 0$; (iv) $x^3 + 3x^2 - 10x - 24 = 0$;
(v) $x^4 - 5x^2 + 4 = 0$; (vi) $x^4 + 4x^3 + x^2 - 6x = 0$;
(vii) $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$; (viii) $x^3 + 3x^2 = 0$;
(ix) $x^4 + x^3 - 30x^2 = 0$; (x) $x^5 - 7x^3 - 6x^2 = 0$.

7. (i) $\pm 2, \pm 3$; (ii) $+1, +9, +1, +9$; (iii) $+1, +16$;
(iv) $+49, +25, +49 + 25$; (v) $+ -1, +3$;
(vi) $(x + 2)^2(x - 2), +2, -2, -2$; (vii) $0, +3, +4$;
(viii) $x^2(x + 8)(x - 5), 0, 0, -8, +5$; (ix) $(x - 1)^2(x - 2)^2, +1, +1, +2, +2$; (x) $(x - 2)^3, +2, +2, +2$;
(xi) $-2, -3, -1, +6$; (xii) $\pm 1, (-1 \pm \sqrt{-3})/2$;
(xiii) $\pm\sqrt{3}, \pm\sqrt{13}$; (xiv) $-1, +4, -2, +5$;
(xv) $(x + 1)^3(2x - 5), -1, -1, -1, +5/2$.

8. (i) $x = +5, -1$; (ii) $x = -2, -7$; (iii) $x = \quad 3, 1/2$;
(iv) $x = -3, 1/3$.

9. (i) $0, -120$; (ii) $0, +48$; (iii) $0, 0$.

11. $y = x^3 - 7x - 6$. 12. Latter = former + 1.

13. $-3, -2, +2$. 14. $0{\cdot}5$ to left; $-1{\cdot}5, +1, +2$.

15. $-4, -2, +2{\cdot}5$; $-0{\cdot}8, +0{\cdot}6, +1{\cdot}2$.

EXERCISE XLV.

See ch. XXIV., § 6; ch. XXVI., D.

A.

1. (i) $S = 160 - 20t + t^2$; (ii) $S = 100 - 21t + 3t^2/2$;
(iii) $S = 192 - 48t + 3t^2$; (iv) $S = 150 - 40t + 5t^2/2$;

The highest points reached are given by (i) $t = 10, S = 60$;
(ii) $t = 7, S = 26{\cdot}5$; (iii) $t = 8, S = 0$; (iv) $t = 8, S = -10$;

In (iii) the marble just reaches top of board while in (iv) it shoots off the board.

At a turning point $v = 0$.

2. (i) $t = 10 \pm \sqrt{S - 60}$; (ii) $t = 7 \pm [(\sqrt{6S - 159})/3]$;
(iii) $t = 8 \pm (\sqrt{3}/3)$; (iv) $t = 8 \pm (\sqrt{10S + 100}/5)$;

The marble crosses middle point of board at times given in seconds by

(i) $t_1 = 10 - 2\sqrt{10}, t_2 = 10 + 2\sqrt{10}$; (ii) $t_1 = 0, t_2 = 14$;
(iii) $t_1 = 8 - (10\sqrt{3}/3), t_2 = 8 + (10\sqrt{3}/3)$;
(iv) $t = 8 - 2\sqrt{11}$.

3. Greatest height of ball = 72 inches at a distance of 4 feet from bowling crease.

Height when ball reaches plane of wickets = − 4·88 inches. Hence ball falls short.

4. $d = 4 \pm \sqrt{3600 - 50h}$ feet where the negative sign is to be taken from $h = 71{\cdot}68$ to $h = 72$ and after this value the + sign is to be taken.

64·05 ft.

The batsman hits ball 7 feet from wicket.

5. Greatest height = 30 feet.

6. $d = 99 \pm \sqrt{1000\,(30 - h)/3}$, where the − sign is taken from $h = {\cdot}597$ feet to 30 feet and after this value the + sign is taken.

The fieldsman is on the circumference of one of two circles of radius 188·4 feet and 9·6 feet respectively from wicket.

If ball had not been caught its range would = 199 feet from wicket.

7. $S = 30n - n^2$; S is maximum when $n = 15$.

8. $n = 15 \pm \sqrt{225 - S}$;
(i) $n = +10, +20$; (ii) $n = -2, n = +32$.

9. (i) $n = (17 \pm \sqrt{289 - 4S})/2$; (ii) $n = 8, 9$. The 9th term is 0.

10. (i) 72; (ii) S is an integer.

11. $S = 70, n = 7$.

12. (i) 12; (ii) $n = -1$ gives − 22 and this is − the value of the sum obtained by continuing the series in opposite direction.
$n = \frac{5}{3}$; $n = \frac{44}{3}$, are neglected.

13. $n = 21$ terms; $S_{21} = -861$.

14. $n = (83 \pm \sqrt{6889 + 8S})/4$;
(i) $n = 3$; (ii) $n = 30$; (iii) $n = 42$;
(iv) If $n = -2$ − $S_{-2} = 174$.

B.

16. (i) $y = (x + 2)/3$; (ii) $y = (2{\cdot}8 - x)/{\cdot}7$;
(iii) $7y - 4x + 8 = 0$; (iv) $3y + 12x - 5 = 0$;
(v) $y = 2 + 3/x$; (vi) $y = 2/x - 7 + 3/2$;
(vii) $y = (15x + 21)/6x + 8$; (viii) $8xy + 2y - 20x - 5 = 0$;
(ix) $x = 3y^2 + 7y - 1$; (x) $x = -2y^2 + y - 8$.

17. $x = ay + b$.

19. (i) $y > -\frac{61}{12}$; (ii) $y < -63/8$. For inverse functions write x for y.

20. (i) $x = 2y^2 - 3y + 4$; (ii) $x = 7 + 3y - 5y^2$;
(iii) $x = ay^2 + by + c$; (iv) $y = \dfrac{4x}{3x - 2}$;
(v) $y = \dfrac{cx}{a - bx}$; (vi) $y = (x \pm \sqrt{x^2 - 3x})/3$;
(vii) $y = (6x \pm \sqrt{8x - 31x^2})/4x$;

(viii) $y = (2x + 1 \pm \sqrt{1 + 4x})/2x$; (ix) $y = (x^2 - 3)/2$;

(x) $y = \left(\dfrac{x^2 + 2}{2x}\right)^2$.

22. (i) x cannot lie between 0 and 5 ;
(ii) x cannot lie between -3 and $+\frac{1}{2}$;
(iv) x cannot lie between -7 and $+\frac{3}{2}$;
(v) $x > 3$ or < -7 ;
(vi) $x < -2$ or $\frac{8}{3} > x > 2$. No limits for values of function.

24. (i) Negative in both cases ;
(i) $x = 0$ is an upper turning-point ;
(ii) $x = 1$ is a lower turning-point.

26. $x = -1$ gives a lower turning-point. The inverse function has no finite turning-points.

27. (i) $(x = 0, y = -3)$; $(x = 0, y = \frac{1}{2})$ are respectively lower and upper turning points.
(ii) $(x = 0, y = 7)$, $(x = 0, y = \frac{3}{2})$ are turning-points.

28. $y = (7 + 3x^2)/(x^2 - 1)$. The inverse function has turning values $(x = 0, y = -7)$.

29. $y = [-3x^2 \pm \sqrt{9x^4 + 32x^2 + 16}]/2$.

30. Upper turning val. $= 8/31$ when $x = 3/4$.

EXERCISE XLVI.

See ch. XXIV., § 6.

A.

3. (i) $b = c \cos \alpha \pm \sqrt{(a^2 - c^2 \sin^2 \alpha)}$; (ii) $\sin \alpha$ not $> (a/c)^2$; (iii) two ; (iv) $\sin \alpha = a/c$.

4. $148\frac{3}{4}$ yards, $29\frac{1}{2}$ yards. 5. 1·6 or 0·8 miles approx.

7. 11·4 or 129·6 ; $B = 94°$ or $13°$; $C = 50°$ or $130°$.

9. $OP \,.\, OP^1 = d^2 - r^2$.

13. $\cos \alpha = \pm \sqrt{\{1 - (5/13)^2\}}$, etc. $\alpha = 22° 37'$ or $157° 23'$.

14. $14° 15'$ or $194° 15'$. 15. $323° 8'$.

EXERCISE XLVII.

See ch. XXIV., § 7 ; ch. XXVII., A.

A.

1. (i) $\dfrac{2x^3}{3}$; (ii) $\cdot 6x^3$; (iii) $-\cdot 3x^3$;
(iv) $4x^{\frac{3}{2}}$; (v) $x^{\frac{3}{2}}$; (vi) $1{\cdot}7(-x)^{\frac{3}{2}}$.

2. 144 ; 300 ; 500 ; 1700 ; 307·2.

3. $480\frac{2}{3}$. 4. 729·6. 5. 335.

6. (A) 542·9 ; (B) $66\frac{2}{3}$; (C) 413·2.

7. $296\sqrt{3}/3 = 171{\cdot}2$. 8. $76\sqrt{10}/3 = 80$.

9. ·04 ; $106\frac{2}{3}$. 10. $266\frac{2}{3}$. 11. $\cdot 04x^2 + \frac{4}{5}x$.

12. (i) $\frac{3}{2}x^2 + \cdot 7x^3$; (ii) $\cdot 8x^2 + \cdot 3x^3$; (iii) $12x + 2x^2 + \dfrac{x^3}{3}$;
(iv) $10\cdot 8x - 1\cdot 2x^2 + \cdot 5x^3$; (v) $\frac{2}{3}x^3 - \frac{3}{2}x^2 + 7x$;
(vi) $\dfrac{x^3}{6} + \dfrac{5}{2}x^2 - 8x$; (vii) $x + 4x^{\frac{3}{2}}$; (viii) $2\cdot 6x^2 - 3x^{\frac{3}{2}}$.
(ix) $\cdot 5x^3 - \cdot 8x^{\frac{3}{2}}$; (x) $\cdot 6x - \cdot 8x^{\frac{3}{2}} + 1\cdot 2x^2 - 1\cdot 6x^3$.
13. 1499·3; 100; 2656; 1347.
14. $y = 20 + 4x - \frac{1}{4}x^3$; $533\frac{1}{3}$.

B.

15. $\dfrac{2n+1}{6n}$; 385π.
16. Volume of cone $= \frac{1}{3}\pi r^2 h$.
17. $\pi h^2 \tan^2 35°$; $\frac{1}{3}\pi h^3 \tan^2 35°$; $\frac{1}{3}\{(3 + 2h)^2(h + 1\frac{1}{2}) - 13\frac{1}{2}\}$; $2023\frac{1}{3}$.
18. $3 + 2h$; $(3 + 2h)^2$.
20. $\pi(12 - \sqrt{3}x)^2$; $\pi(144x - 48x^{\frac{3}{2}} + \frac{9}{2}x^2)$; 384π.
24. $\dfrac{3k}{4} \times \frac{4}{3}$.
26. (i) $\cdot 6x^4$; (ii) $-\cdot 8x^4$; (iii) $\cdot 8x^{\frac{3}{2}}$; (iv) $-6x^{\frac{4}{3}}$; (v) $x - \frac{1}{2}x^4$;
(vi) $x + x^2 + x^3 + x^4$; (vii) $3\cdot 2x - \cdot 7x^2 + \cdot 9x^3 - \cdot 8x^4$;
(viii) $4x - 9x^{\frac{4}{3}}$; (ix) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$; (x) $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4}$;
(xi) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$; (xii) $8x + 6x^2 + 6x^3 + \frac{27}{4}x^4$.
27. 777·6; 8000; $120\sqrt{10}$; 136; $93\frac{1}{3}$; $244\frac{1}{3}$.

EXERCISE XLVIII.

See ch. XXIV., § 7; ch. XXVII., B.

A.

1. (i) $3h$, $\delta y/\delta x = 3$; (ii) $(a - h)h$; $\delta y/\delta x = (a - h)$;
(iii) $4\cdot 1(2xh + h^2)$; $\delta y/\delta x = 8\cdot 2x$;
(iv) $4\cdot 1(2xh + h^2) - 3\cdot 7h$; $\delta y/\delta x = 8\cdot 2x - 3\cdot 7$;
(v) $7(3x^2h + 3xh^2 + h^3)$; $\delta y/\delta x = 21x^2$;
(vi) $-h/2x^2 + 2hx$; $\delta y/\delta x = -1/2x^2$.
2. (i) $\delta y/\delta x = 1/4$; (ii) $\delta y/\delta x = -3\cdot 5$; (iii) $\delta y/\delta x = 5\cdot 4x$;
(iv) $\delta y/\delta x = 8\cdot 6x - 1\cdot 2$; (v) $\delta y/\delta x = 12 - 10x$;
(vi) $\delta y/\delta x = 6x^2$; (vii) $\delta y/\delta x = 3x^2 - 4$;
(viii) $\delta y/\delta x = 12x^2 + 2x - 7$; (ix) $\delta y/\delta x = 3 - 1/3x^2$;
(x) $\delta y/\delta x = 5 + 7/x^2$.
(i) $y = 2x + p$; (ii) $y = -\frac{1}{3}x + p$; (iii) $y = 4x^2 + p$;

3. (iv) $y = x^2 - 3x + p$; (v) $y = 5x - \frac{3}{2}x^2 + p$;
(vi) $y = 2x^3 - 2x^2 + x + p$; (vii) $y = -3/x + p$;
(viii) $y = x + 1/x + p$.
4. (i) 19 ; (ii) $\frac{4}{3}$; (iii) 0 ; (iv) -5 ; (v) 1 ;
(vi) 0 ; (vii) 2 ; (viii) 0.
5. (i) $\delta^2y/\delta x^2 = \frac{2}{5}$; (ii) $\delta^2y/\delta x^2 = 30x - 6$;
(iii) $\delta^2y/\delta x^2 = 12(x - 1)$.
6. (i) $y = -\frac{3}{2}x^2 - 5x + 1$; (ii) $y = x^3 - x^2 + x - 1$;
(iii) $y = 2x^3 - 11x^2$.
8. (i) $y = \frac{1}{4}ax^4 + p$; (ii) $y = \frac{1}{6}x^4 - \frac{1}{2}x^3 + \frac{1}{2}x^2 + px + q$;
(iii) $y = \frac{5}{24}x^4 + px^2 + qx + r$;
(iv) $y = -\frac{1}{24}x^4 + px^3 + qx^2 + rx + s$.

B.

11. (i) 2×10^{-5} ; (ii) $\frac{1}{5}$; (iii) 2×10^{-7} ; (iv) 2×10^{-10}.
13. 6×10^{-4}.
15. (i) $\frac{2}{3}$; (iii) 1·5 ; (iv) 2, -3 ; (v) $\frac{5}{2}$, 3.
16. (iv) $-\frac{1}{2}$; (v) $\frac{11}{4}$.

EXERCISE XLIX.

See ch. XXIV., § 7.

A.

1. (i) 9 ; (ii) 0, max. ; (iii) -3 ; (iv) 0, min. ; (v) 9.
3. (i) $\frac{33}{2}$; (ii) 0, max. ; (iii) -36 ; (iv) -36 ; (v) -24 ;
(vi) 0, min. ; (vii) $\frac{33}{2}$.
6. $q^2 > 3pr$.
7. (i) $\frac{2}{3}, \frac{20}{3})$; (ii) $(-·25, 7·3675)$;
(iv) $(-\frac{1}{2}, \frac{63}{4})$, $(3, -25)$; (v) $(0, -1)$.

B.

8. (ii) Turning-point ; (iii) point of inflexion ; (iv) turning-point.
9. (ii) Turning-point ; (vi) turning-point.
10. Where $x = \frac{4}{3}$, $x = 2$.
11. (iii) Where $x = -1$; (iv) Where $x = \frac{5}{4}$.

C.

19. No max. Min. = 486 cubic inches. 21. 10.
22. 2 feet × 1 foot × 1 foot.

EXERCISE L.

See ch. XXIV., § 9 ; ch. XXVIII., A and B.

A.

2. $C = 2$, $C_1 = \sqrt{2}$. 4. (i) 2 ; (ii) 2r. 6. 1·158.
7. $3 \angle \pi \angle 3·474$. 9. $3·103 \angle \pi \angle 3·215$.
10. $8\sqrt{(2 - \sqrt{2})}$, $16\sqrt{\{(2 - \sqrt{2})/(2 + \sqrt{2})\}}$.

11. $3{\cdot}1012 \angle \pi \angle 3{\cdot}3136$.
12. $C = \sqrt{\{(4 - C,)C,\}}$. 13. $\sqrt{3}$.
14. $2\sqrt{3}$; $2{\cdot}598 \angle \pi \angle 5{\cdot}196$.

B.

17. $\sin \frac{a}{2} = \frac{\sqrt{2}}{2} \{1 - \sqrt{(1 - \sin^2 a)}\} = \frac{\sqrt{2}}{2} (1 - \cos a)$.

18. $0{\cdot}065$.

SECTION III.

EXERCISE LI.

See ch. XXIX., § 2.

A.

1. 2·1 inches; 2·1 inches; 2·1 inches; 1·048; 1·038; 1·040; the ratio.
2. 1·052; (i) 58·3 inches; (ii) 63·1 inches.
3. 58·2 inches.
4. Boy: 1·052, 1·038, 1·033, 1·02.
   Girl: 1·048, 1·040, 1·043, 1·020.
5. 51·7, 53·8, 55·9, 58·1, 60·4, 62·9 inches.
6. 44·2 inches, 42·5 inches.
7. 1·02. 8. 50719; 46857. 9. 0·99. 10. 4386; 4566.
12. $P = P_0 \times (1{\cdot}02)^n$; $P = P_0 \times (0{\cdot}99)^n$.
13 and 14. The factors are given in the following table:—

Time.	− 5	− 4	− 3	− 2	− 1	0	+ 1	+ 2	+ 3	+ 4	+ 5
r = 1·1	·620	·683	·751	·826	·909	1·000	1·100	1·210	1·331	1·464	1·610
r = 1·3	·268	·349	·454	·591	·769	1·000	1·300	1·690	2·197	2·856	3·712
r = 0·8	3·050	2·440	1·592	1·562	1·250	1·000	·800	·64	·512	·409	·327

B.

15. $\sqrt{2} = 1{\cdot}414$. 16. $\sqrt[4]{2} = 1{\cdot}189$.
17. Growth factor constant for all equal intervals.
18. (i) 1·3; (ii) 7237; (iii) 3371.
19. 1·3. 20. 0·8; 630.
21. 51·8, 53·8, 55·8, 57·8 inches.

EXERCISE LII.

See ch. XXIX., § 2; ch. XXX.

A.

2. 25%. 3. (i) 69·80 ounces; (ii) 74·97 ounces; (iii) 35·82 ounces.
4. 2629. 5. 15822; 33861.

6. (i) £1 5s. 3d., £1 18s. 6d., 11s. 5d., £1 14s. 11d. ;
(ii) £1 1s. 6d., £1 4s. 4d., 16s. 11d., £1 3s. 8d. ;
(iii) 14s. 8d., 8s. 6d., £2 15s. 4d., 9s. 8d. ;
(iv) 6·44 years time, 1·64 years time, 4·11 years ago, 2·11 years time.
7. (i) 182·8 fathoms ; (ii) 257·0 fathoms ; (iii) 300·9 fathoms ; (iv) 83·23 fathoms ; (v) 62·37 fathoms ; (vi) 29·54 fathoms.
8. (i) 878·6 fathoms ; (ii) 68·16 fathoms.
9. (i) 26·4 ; (ii) 74·36 ; (iii) 19·46 ; (iv) 22·03.
10. (i) £2 18s. 5d. ; (ii) £3 8s. 3d. ; (iii) £6 10s. 6d. ; (iv) £2 2s. 8d. ; (v) £1 3s. 11d.
11. (i) 3451 ; (ii) 2852 ; (iii) 4046 ; (iv) 4243.
12. (i) 2925 ; (ii) 2430.
13. (i) 1·3 ; (ii) 30°/ₒ ; (iii) (*a*) 5·29 years ago, (*b*) 6·14 years ago.
14. (i) 21·21°/ₒ (ii) 42·76°/ₒ.
15. The first one ; 30°/ₒ, 25°/ₒ.

B.

16. (i) 51 ; (ii) 72 ; (iii) 130 ; (iv) 7 years ago.
17. (i) 54 ; (ii) 83 ; (iii) 4·6 years ago.
18. (i) 152 ; (ii) 395 ; (iii) 283.
19. 214. 20. 14°/ₒ ; 11·8°/ₒ.

EXERCISE LIII.

See ch. XXIX., § 3 ; ch. XXXI.

A.

1. 1·26 ; 1·52 ; 3·5. 2. x = 1·26t ; x = 1·52t ; x = 3·5t.
3. 3·09. 4. (i) 1·09 ; (ii) 1·09.
5. (i) 8·82 ; (ii) 8·06 ; (iii)
6. 6 hrs. approx. 7. (i) 166 ; (ii) 250.
8. (i) 3·24 inches ; (ii) 3·6 inches ; (iii) 2·67 inches.
9. (i) 1·21 ; (ii) 1·22 ; (iii) 0·89. 10. 1·54 11. 1·16.
12. (i) $\frac{5x}{8}$; (ii) $\frac{5x}{8}$; (iii) $-\frac{11x}{6}$; (iv) $\frac{px}{q}$; (v) $-\frac{3px}{q}$; (vi) $q\frac{ax}{p}$.

B.

13. (i) 4·82 ; (ii) 7·3 ; (iii) 4·71 ; (iv) 2·61 ; (v) 2·86 ; (vi) 1·43 ; (vii) 1·69 ; (viii) 1·77 ; (ix) 1·79 ; (x) 1·79 ; (xi) 5·75 ; (xii) 5·75 ; (xiii) 2·32 ; (xiv) 3·4 ; (xv) 1·66.
14. (i) 3·32 ; (ii) 2·46 ; (iii) 1·53 ; (iv) 1·37 ; (v) 5·89 ; (vi) 8·43.
16. (i) 38·7 ; (ii) 8·52 ; (iii) 32 ; (iv) ·144.
17. (i) 59·8 ; (ii) ·198 ; (iii) 2·8 ; (iv) 17·25.
18. (i) 2·1 ; (ii) 1·46 ; (iii) 13·1.
19. (i) 1·53 ; (ii) 4·41 ; (iii) 20·7 ; (iv) ·36
20. 12·9. 22. (i) 1·78 ; (ii) 0·46.

EXERCISE LIV.

See ch. XXIX., §§ 3, 4; ch. XXXII., A and B.

A.

1. (i) 3·52; (ii) 1·375; (iii) 0·73; (iv) 0·33.
2. (i) 1·25; (ii) 1·04; (iii) 1·02.
4. (i) 9·18; (ii) 1·26; (iii) 1·35; (iv) 7·32.
5. (i) 9180; (ii) 0·000129; (iii) 0·00000945; (iv) 128.
6. (i) 4·11; (ii) 4·28; (iii) 4·28.
8. (i) 3·71; (ii) 5·46; (iii) 5·69; (iv) 6·58; (v) 3·69; (vi) 4·75.
9. (i) 1·47; (ii) 1·89; (iii) 6·58.
10. (i) 4·50; (ii) 498; (iii) 2·97; (iv) 155000; (v) 431.
11. (i) 0·422; (ii) 0·442; (iii) 0·663; (iv) 7·12; (v) 0·0821.
12. (i) 14·62; (ii) 50·82; (iii) 16560; (iv) 0·0003654; (v) 630900.

B.

13. Logarithms for base 1·25.

n	log n	n	log n	n	log n	n	log n
0·5	− 3·11	0·9	0·47	1·3	1·18	1·7	2·37
0·6	− 2·29	1·0	0·00	1·4	1·51	1·8	2·63
0·7	− 1·60	1·1	0·43	1·5	1·82	1·9	2·88
0·8	− 1·00	1·2	0·82	1·6	2·11	2·0	3·11

14. Anti-logarithms for base 1·25.

l	antilog l	l	antilog l	l	antilog l	l	antilog l
− 3·0	0·51	− 1·4	0·73	0·2	1·05	1·8	1·49
− 2·8	0·54	− 1·2	0·76	0·4	1·09	2·0	1·57
− 2·6	0·56	− 1·0	0·80	0·6	1·14	2·2	1·63
− 2·4	0·59	− 0·8	0·84	0·8	1·19	2·4	1·71
− 2·2	0·61	− 0·6	0·87	1·0	1·25	2·6	1·79
− 2·0	0·64	− 0·4	0·91	1·2	1·31	2·8	1·87
− 1·8	0·67	− 0·2	0·96	1·4	1·37	3·0	1·95
− 1·6	0·70	0·0	1·00	1·6	1·43		

15. (i) − 1·93; (ii) 0·93; (iii) 2·43.
16. (i) 1·8; (ii) 1·2; (iii) 0·7.
17. (i) 1·17; (ii) 0·88; (iii) 0·14.

18. (i) 0·69 ; (ii) 1·25 ; (iii) 1·85 ; (iv) 0·58.

19. (i) $\log_a$ PQ ; (ii) $\log_a \frac{P}{Q}$; (iii) n $\log_a$ P ; (iv) $\frac{1}{n}\log_a$ P.

20. (i) antilog_a (P + Q) ; (ii) antilog_a (P − Q) ;
(iii) antilog_a nP ; (iv) antilog $\frac{P}{n}$.

EXERCISE LV.

See ch. XXIX., § 4 ; ch. XXXIII., A.

A.

1. (i) 1·45 ; (ii) 2·87 ; (iii) 3·82 ; (iv) 4·54 ; (v) 5·09 ; (vi) 5·68.
2. (i) 1·23 ; (ii) 2·54 ; (iii) 3·38 ; (iv) 6·73 ; (v) 8·24.
3. (i) 2·10 ; (ii) 2·91 ; (iii) 3·32.
4. (i) 1·74 ; (ii) 2·64 ; (iii) 6·96.
5. (i) 1·70 ; (ii) 3·53 ; (iii) 8·34.
6. (i) 2·21 ; (ii) 6·19 ; (iii) 2·91.

B.

7. 1·1, 1·21, 1·33, 1·46, 1·61, 1·77, 1·95, 2·14, 2·36, 2·59.
9. (i) 4·177 ; (ii) 4·595 ; (iii) 6·73 ; (iv) 7·40.
10. 1·61. 11. (i) 1·4 ; (ii) 1·8 ; (iii) 1·46 ; (iv) 2·14 ; (v) 4·59.
12. (i) 0·86 ; (ii) 1·46 ; (iii) 3·93.
13. 2·59. 14. (i) 0·4 ; (ii) 0·9 ; (iii) 0·73 ; (iv) 1·33 ; (v) 7·40.
15. 0·656

16.

antilog n	0·00	0·25	0·50	0·75	1·00	1·25
n	1·0000	0·9000	0·8100	0·7290	0·6561	0·5905
antilog n	1·50	1·75	2·00	2·25	2·50	
n	0·5315	0·4784	0·4306	0·3875	0·3488	

17. (i) 1·75 ; (ii) 0·75 ; (iii) 2·18.
18. (i) 0·5905 ; (ii) 0·4306 ; (iii) 0·7776.
19. (i) 1·46 ; (ii) 1·25 ; (iii) 1·61 ; (iv) 0·125.
20. (i) 0·6 ; (ii) 0·4784 ; (iii) 0·3488.

EXERCISE LVI.

See ch. XXIX., § 6; ch. XXXIII., B.

A.

1. (i) 0·301; (ii) 0·531; (iii) 0·021; (iv) 0·940; (v) 0·968.
2. (i) 3·301; (ii) 6·531; (iii) $\bar{3}$·531; (iv) 4·021; (v) $\bar{1}$·021; (vi) $\bar{5}$·940; (vii) 2·968.
3. (i) 2·51; (ii) 5·62; (iii) 2·33; (iv) 8·81; (v) 1·16.
4. (i) 251; (ii) 0·00251; (iii) 0·562; (iv) 23300; (v) 0·00000233; (vi) 88·1; (vii) 0·0000881; (viii) 116.
5. (i) 0·767; (ii) 1·013; (iii) 0·823; (iv) 1·155; (v) 3·471; (vi) 0·053; (vii) 5·079; (viii) 0·343.
6. (i) 5·85; (ii) 10·30; (iii) 6·65; (iv) 14·29; (v) 2958: (vi) 1·13; (vii) 119900; (viii) 2·20.

B.

7. (i) 0·357; (ii) 0·000935; (iii) 0·000115; (iv) 11·5;
8. (i) 0·0000748; (ii) 0·305; (iii) 0·622; (iv) 0·0542; (v) 1·02.
9. (i) 112·3; (ii) 34810; (iii) 0·000337; (iv) 0·0363.
10. (i) 60·46; (ii) 0·110; (iii) 7·59.

EXERCISE LVII.

See ch. XXIX., § 6; ch. XXXIII., C.

A.

1. (i) 0·63175; (ii) 4·63175; (iii) ^{4}·63175; (iv) 3·30856; (v) 1·30856; (vi) $\bar{1}$·88098; (vii) 2·72444; (viii) 3·00290; (ix) $\bar{3}$·00290; (x) 0·90309; (xi) 4·00000; (xii) $\bar{5}$·00000.
2. (i) 1·7498; (ii) 17498·; (iii) 0·017498; (iv) 8173·4; (v) 1·0778; (vi) 0·0010778; (vii) 0·10109; (viii) 1·0014; (ix) 1001·4; (x) 0·000015849.
3. (i) $\bar{1}$·32319; (ii) $\bar{2}$·31146; (iii) 6·09269; (iv) 3·99710; (v) $\bar{3}$·99710; (vi) $\bar{6}$·55112; (vii) $\bar{1}$·64294; (viii) $\overline{12}$·01160; (ix) 0·92635; (x) $\bar{1}$·32635; (xi) $\bar{1}$·960326; (xii) $\bar{1}$·250725;
4. (i) 0·210475; (ii) 0·020486; (iii) 1237870·; (iv) 9933·5; (v) 0·0099335; (vi) 0·00000355726; (vii) 0·126207; (viii) 0·0000000000010271; (ix) 8·4401; (x) 0·21201; (xi) 0·912706; (xii) 0·17813.
5. (i) 103·2; (ii) 0·000003474; (iii) 0·6812; (iv) 0·3374; (v) ·01721.

B.

6. (i) 62801733 ccs.; (ii) 2·2576 feet.
7. (i) £636 14s. 10d.; (ii) £181 11s. 0d.
8. 97·9 years. 9. 4·51%.
10. (i) 423900000; (ii) 324200000.
11. (i) 4·993816; (ii) 0·00614619; (iii) 0·00003831.

12. (i) 98·28%; (ii) 99·97%.
13. n. 14. n.
15. (i) 31; (ii) 785; (iii) 33.
16. (i) 30; (ii) 784.
17. 10. 19. 10. 20. 2. 21. 2. 23. 9. 24. 16.

C.

25. £636·46. 26. £64·015. 27. £642·6.
28. (i) £545·37; (ii) £652·055.

EXERCISE LVIII.

See ch. XXIX., § 7; ch. XXXIV.

A.

5. (i) 1·57; (ii) 2·44; (iii) 0·54.
6. (i) 0·75; (ii) 0·54; (iii) 0·44.
8. (i) ·78368, 6·077; (ii) $\bar{4}$·64494, 0·0004406; (iii) 5·93542, 861820.
10. (i) 229·1; (ii) 0·000191; (iii) 0·00000316; (iv) 1959·4; (v) 0·1316; (vi) 1273×10^6.
12. (i) 1·28; (ii) 1·38; (iii) 0·64.
13. (i) 0·000007943; (ii) 5·749; (iii) 11·22.
14. (i) $x^{\frac{2}{3}}$; (ii) $x^{\frac{3}{5}}$; (iii) $x^{\frac{5}{7}}$; (iv) $x^{\frac{5}{4}}$.

B.

15. (i) $y = 3x^{\frac{3}{2}}$; 94·86; (ii) $y = 3x^{\frac{2}{3}}$, 13·925, 13·925; (iii) $y = 2x^{\frac{3}{4}}$, 11·247; (iv) $y = 256x^{\ 1}$, 25·6, − 25·6.
17. $y = ax^m$, where a = antilog e. 18. $y = 30{\cdot}2x^{0{\cdot}24}$.
19. (i) $y = 6{\cdot}31x^{0{\cdot}2}$; (ii) $y = 70{\cdot}8x^{-0{\cdot}74}$.
20. $a = 2{\cdot}21$, $m = 0{\cdot}421$. 21. $p = 6{\cdot}46u^{-0{\cdot}795}$

C.

23. $(1 + \frac{1}{n})^n$. 24. (i) 2·5936; (ii) 2·704; (iii) 2·716; (iv) 2·717.
28. (i) $a^{\frac{2}{3}} b^{-1}$; (ii) $a^{-1\frac{3}{4}} b^{\frac{4}{5}}$; (iii) $(x^{\frac{4}{3}} - y^{\frac{4}{3}})$; (iv) $x - y^{\frac{1}{2}}$.

EXERCISE LIX.

See ch. XXIX., § 8.; ch. XXXV.

A.

1. (i) 0·0506, 5·06; (ii) 0·05097, 5·097; (iii) 0·05111, 5·111.
2. The former, with an effective interest of 2·5156 per cent per annum is more profitable than the latter with an effective interest of 2·269 per cent per annum.

3. A has £12·7 more.
5. 20. 6. 6. 7. $V = P(1 + j/p)^{-pt}$.

B.

9. (i) 3·56 ; (ii) 5·12. 11. 5·00.

12. $P = \frac{a}{j}(1 - e^{-jn})$. 13. (i) 10·25 ; (ii) 10·3813 ;
(iii) 10·4718 ; (iv) 10·5093.

SUPPLEMENTARY EXERCISES.

EXERCISE LX.

See ch. XXXVI., § 2.

A.

1. — 4. Oral.

5. (i) $5{\cdot}932$; (ii) $-{\cdot}1148$; (iii) $103{\cdot}35$; (iv) $2{\cdot}924$; (v) ${\cdot}0154$.

6. (i) $1{\cdot}01940$; (ii) $\bar{1}{\cdot}87778$; (iii) $2{\cdot}26276$; (iv) ${\cdot}93366$.

B.

8. (i) $\gamma = 77°$, $b = 144$, $c = 158{\cdot}9$;
(ii) $\gamma = 59° \, 18'$, $b = 16{\cdot}98$, $c = 18{\cdot}19$;
(iii) $\gamma = 44°$, $b = 93{\cdot}6$, $c = 188{\cdot}02$;
(iv) $\gamma = 109° \, 49'$, $b = 712{\cdot}11$, $c = 742{\cdot}58$.

9. (i) $C = 55°$, $b = 498{\cdot}5$ yards, $a = 410{\cdot}77$ yards;
(ii) $C = 46°$, $b = 816{\cdot}97$ feet, $a = 1376{\cdot}56$ feet;
(iii) $A = 13° \, 57'$, $a = 401{\cdot}83$ yards, $c = 1408{\cdot}9$ yards.

12. $28° \, 58'$, $104° \, 28'$, $46° \, 34'$.

14. (i) $\tan \frac{a}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$;

(ii) $\sin a = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

16. (i) $a = 32° \, 35'$, $\beta = 37° \, 24'$, $\gamma = 110° \, 1'$, $\triangle = 1170$;
(ii) $a = 58° \, 57'$, $\beta = 67° \, 55'$, $\gamma = 53° \, 18'$, $\triangle = 16619$;
(iii) $a = 16° \, 48'$, $\beta = 113° \, 56'$, $\gamma = 49° \, 16'$, $\triangle = 669{\cdot}7$;
(iv) $a = 85° \, 10'$, $\beta = 39° \, 10'$, $\gamma = 55° \, 40'$, $\triangle = 37951$.

EXERCISE LXI.

See ch. XXXVI., § 3.

C.

21. (i) $\beta = 104° \, 51'$, $\gamma = 33° \, 9'$, $a = 19{\cdot}8$;
(ii) $\beta = 30° \, 17'$, $\gamma = 43° \, 13'$, $a = 397{\cdot}2$;
(iii) $a = 93° \, 37'$, $\beta = 36° \, 3'$, $c = 121{\cdot}09$;
(iv) $a = 24° \, 12'$, $\gamma = 33° \, 48'$, $b = 1250{\cdot}6$;
(v) $a = 167° \, 32'$, $\gamma = 4° \, 2'$, $b = 83{\cdot}12$.

22. (i) There is no triangle satisfying the conditions.
(ii) $\alpha = 25° 18'$, $\gamma = 116° 42'$, $c = 298{\cdot}92$;
(iii) $\alpha = 33° 33'$, $\beta = 50° 27'$, $a = 170{\cdot}61$.
(iv) There are two possible triangles :—
$\alpha = 90°$, $\gamma = 60°$, $a = 22\sqrt{3}$.
$\alpha = 30°$, $\gamma = 120°$, $a = 11\sqrt{3}$.

EXERCISE LXII.

See ch. XXXVI., § 4.

6. (i) $2 \cos 30° . \cos 7°$; (ii) $2 \sin 30° . \cos 7°$;
(iii) $2 \cos 60° . \cos 12°$; (iv) $2 \sin 80\tfrac{1}{2}° . \cos 23\tfrac{1}{2}°$;
(v) $2 \sin 58° . \sin 24°$; (vi) $2 \cos 70° . \sin 54°$;
(vii) $2 \sin 55° . \sin 73°$; (viii) $2 \cos 55° . \sin 73°$.
7. (i) $\cos 47° + \cos 17°$; (ii) $\cos 52° + \cos 58°$;
(iii) $\sin 70° + \sin 16°$; (iv) $- \sin 1° - \sin 75°$;
(v) $- \tfrac{1}{2}(\sin 31° + \sin 35°)$; (vi) $14(\cos 37° - \cos 71)$.
8. (i) $\tan 52°$; (ii) $\cot 52\tfrac{1}{2}°$; (iii) $\tan 4\alpha$, ; (iv) $\tan 2\alpha$;
(v) $\tan (\alpha - \beta)$; (vi) $\tan \tfrac{1}{2}(\alpha + \beta)$.
11. (i) $\cos 2\alpha$; (ii) $\tan \tfrac{1}{2}\alpha$; (iii) $\cot \tfrac{1}{2}\alpha$; (iv) $\cos \alpha + \sin \alpha$;
(v) $\sin 2\alpha$; (vi) $\tan 2\alpha$; (vii) $\tan^2\alpha$; (viii) $\tan \alpha$.

EXERCISE LXIII.

See ch. XXXVI., § 5.

A.

1. (i) 48·42 feet, 4·035 seconds ; (ii) 57 feet, 4·75 seconds ;
(iii) 81 feet.
2. (i) 54·5, − 3·19 ; (ii) 54·53 feet.
3. (i) 55·49 feet ; (ii) 4·59 seconds.
4. (i) $d = 20t$, $h = 4d - \dfrac{d^2}{25}$;
(ii) $2\tfrac{1}{2}$ seconds, at height of 100 feet.
(iii) Range is 100 feet.
5. 97·98 feet, 4·875 seconds. 6. 103 feet, 5·122 seconds.
7. $y = \dfrac{a^2(2c - 1)}{4bc^2}$; $x = \dfrac{a}{2b}$; $t = \dfrac{a}{2hc}$.
8. $x = 1{\cdot}75$.
9. (i) $y = \dfrac{2(x + 3)^2}{2 - 3(x + 3)}$;
(ii) $x = -5$, $z = -1$ or $x = -\tfrac{5}{2}$, $z = \tfrac{1}{4}$.

EXERCISE LXIV.

See ch. XXXVI., § 5.

A.

1. $x^2 + y^2 + 6x - 8y + 9 = 0$.
4. (i) Circle : centre $(0 \cdot 0)$, radius 5 ;
 (ii) Circle : centre $(0 \cdot 0)$, radius $\frac{4}{3}$;
 (iii) Circle : centre $(2 \cdot {}^{-3})$, radius 5 ;
 (iv) Circle : centre $(-7 \cdot 4)$, radius 8 ;
 (v) Circle : centre $(-5 \cdot {}^{-12})$, radius 13 ;
 (vi) Circle : centre $(\frac{7}{6} \cdot 0)$, radius $\frac{5}{6}$;
 (vii) Circle : centre $(0 \cdot \frac{5}{4})$, radius $\frac{5}{4}$;
 (viii) Circle : centre $(a - b)$, radius $\sqrt{3a^2 + 2ab + 3b^2}$.
8. (i) No ; (ii) No ; (iii) $x = 5$, $y = -2$ (only) ;
 (iv) $x = -4$, $y = 6$ (only).
9. (i) $x = -2{\cdot}45$ or $\cdot 65$; $y = 4{\cdot}35$ or $4{\cdot}95$;
 (ii) $x = 4{\cdot}85$ or $-2{\cdot}9$; $y = 1{\cdot}05$ or $4{\cdot}05$;
 (iii) $x = -3$, $y = 4$;
 (iv) $x = 1{\cdot}85$ or $4{\cdot}95$, $y = 4{\cdot}65$ or $\cdot 70$.
10. (i) $x = -2{\cdot}85, -\cdot 35$; $y = -11{\cdot}55, -5{\cdot}40$;
 (ii) $x = -2{\cdot}87, 4{\cdot}87$; $y = -4{\cdot}08, 1{\cdot}08$.
13. (i) $2x - y = 0$; (ii) $3x + 4y = 0$; (iii) $x = 1$.
14. (i) $x = \pm \dfrac{2}{\sqrt{5}}$; $y = \pm \dfrac{4}{\sqrt{5}}$;
 (ii) $x = 3{\cdot}84$ or 0 ; $y = -2{\cdot}88$ or 0 ; (iii) $x = 1$; $y = -1$.
15. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

20. (i) Ellipse : centre $(0 \cdot 0)$, axes 6 and 2 ;
 (ii) Ellipse : centre $(0 \cdot 0)$, axes 10 and 6 ;
 (iii) Ellipse : centre $(0 \cdot 0)$, axes 1 and $\frac{2}{3}$;
 (iv) Ellipse : centre $(0 \cdot 0)$, axes $\frac{1}{2}$ and 2 ;
 (v) Ellipse : centre $(0 \cdot 0)$, axes $2\sqrt{3}$ and $2\sqrt{2}$;
 (vi) Ellipse : centre $(0 \cdot 0)$, axes $2\sqrt{\frac{10}{7}}$ and $2\sqrt{\frac{10}{3}}$.
21. (i) $y = \pm\frac{1}{5}\sqrt{225 - 9x^2}$; x lies between -5 and $+5$;
 (ii) $y = \pm\sqrt{1 - 16x^2}$; x lies between $-\frac{1}{4}$ and $+\frac{1}{4}$;
 (iii) $y = \pm\frac{1}{3}\sqrt{6 - 2x^2}$; x lies between $-\sqrt{3}$ and $+\sqrt{3}$.
22. $x^2 + 9y^2 - 4x + 18y + 4 = 0$.
23. (i) $2x^2 + 3y^2 + 12x - 12y + 24 = 0$;
 (ii) $7x^2 + 3y^2 - 14x - 6y = 0$: (passes through the origin).
24. (i) Axes $y = -3$, $x = 1$. Lengths 4 and 2 ;
 (ii) Axes $y = 3$, $x = -2$. Lengths 4 and 6 ;
 (iii) Axes $y = -1$, $x = 7$. Lengths $2\sqrt{\frac{5}{2}}$ and $2\sqrt{\frac{5}{3}}$.
26. (i) $x = 3$ or 0, $y = 0$ or -2 ;
 (ii) $x = -1{\cdot}67$ or $-2{\cdot}83$, $y = 1{\cdot}66$ or $-0{\cdot}66$;
 (iii) $x = 3$ or 1, $y = -3$ or -2.

EXERCISE LXV.

See ch. XXXVI., § 5.

A.

9. α between 45° and 135°.

11. $\frac{x^2}{9} + \frac{y^2}{16} = 1$; $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

13. (i) 4 ; 90° ; (ii) 4, 2 ; $2 \tan^{-1} \frac{1}{\sqrt{2}}$;
(iii) 6, 4 ; $2 \tan^{-1} \frac{2}{3}$; (iv) $2\sqrt{2}$, $2\sqrt{3}$; $2 \tan^{-1} \sqrt{\frac{2}{3}}$.

14. x not between $-p + a$, and $-p - a$.
y not between $-q + b$, and $-q - b$.

B.

16. $-x = p^2y^2$; $x = p^2y^2$; $-y = p^2x^2$.

17. $(10, -5)$.

18. $x^2 + y^2 - 20x + 10y + 109 = 0$; $(5, -10)$; $(10, \quad 5)$.

19. $\sqrt{3}y - x = -(\sqrt{3}x + y)^2$; $\sqrt{3}(y + 10) - x + 3 = -[\sqrt{3}(x - 3) + y + 10]^2$.

20. $5x^2 + y^2 + 2xy = 8$; $5x^2 + y^2 + 36x + 4y + 2xy = 0$.

21. $13x^2 + 28xy - 8y^2 = 6$; $13x^2 - 56x + 28xy + 100y + 27 = 0$.

22. $8x^2 - 8y^2 + 12xy = 62{\cdot}5$.

27. $\tan 2\alpha = -\frac{3}{4}$.

30. Straight lines.

EXERCISE LXVI.

See ch. XXXVII., § 1.

A.

1. $-{\cdot}13$ feet. 3. 0 ; 1·2 feet ; $-{\cdot}73$ feet.
4. ·013, ·025 feet. 5. ·028 feet. 6. (·11, ·13) feet ; ·17 feet.

B.

7. nkh ; nkx ; $\frac{1}{2}$nkh(nh + 1).

12. $\frac{2}{3}$h from apex along axis.

13. $\frac{3}{4}$h from apex along axis.

15. Mid-way between pole of spherical surface and plane of section.

16. $\frac{3}{4}$h from apex along axis.

17. (i) (3x + 8) inches ;
(ii) On line joining centres of parallel sides, at a perpendicular distance of 5·42 inches from the shorter.

19. $\bar{x}_2 = \frac{\bar{X}(A_1 + A_2) - A_1\bar{x}_1}{A_2}$; $\bar{y}_2 = \frac{\bar{Y}(A_1 + A_2) - A_1\bar{y}_1}{A_2}$;

20. $\dfrac{\left[\frac{h^2}{3} + ah + \frac{a^2}{2}\right]}{\frac{h}{2} + a}$ from vertex, along perpendicular to base.

21. $\dfrac{3a^2 - h^2}{6a - 3h}$ from the edge of square cut away.

22. On line joining centres 2·5 inches from centre of larger circle.
23. On line joining centres 13·5 inches from point of contact.

24. (i) $\pi\,(10 + \frac{x}{3})^2$; (ii) 28·7 inches along axis from smaller surface.

25. Fig. 11 : $\sqrt{\dfrac{2c(2b + a) + b^2}{2(a + b)}}$ from upper edge ; $a/2$ from left edge.

Fig. 12 : $\dfrac{4c(a + 3) + 3b}{2(2a + 3b)}$ from upper edge ;

$\dfrac{2a(a + 2b) - 3bc}{2(2a + 3b)}$ from left edge.

C.

27. $2\pi x \,.\, \delta l$ (i.e. frustum of a cone). 30. $2\pi x \times 4h^2 = 2\pi x \,.\, \delta A$.
32. (i) $A = 4\pi^2(R - r)r$; (ii) $V = 2\pi^2(R - r)r^2$.
33. A = 620 square inches ; V = 930 cubic inches.
34. A = $3323\frac{1}{21}$ square inches V = $7542\frac{6}{7}$ cubic inches.
35. $A = 2\pi r(2a + \pi b)$; $V = 2\pi rb(a + \pi b/4)$. 36. 100·8 pounds.

EXERCISE LXVII.

See ch. XXXVII., § 2.

A.

1. ·867. 2. (i) ·72 feet ; (ii) ·65 feet.
3. ·97 feet. 4. 1·0 feet.
5. (i) A, 5·4 ; B, 5·46 ; (ii) A, 5·105 ; B, 4·912.
B is the best.

B.

7. $\dfrac{1}{\sqrt{3}}$. 8. $\dfrac{1}{2\sqrt{3}}$. 9. $\dfrac{2}{\sqrt{3}}$ feet. 10. (i) $\dfrac{2r}{\sqrt{3}}$; (ii) $\dfrac{r}{\sqrt{3}}$.

12. $\dfrac{h}{\sqrt{2}}$. 13. 5·95 inches. 15. $r\sqrt{\left(\dfrac{1}{2} + \dfrac{1}{2m}\right)}$. 16. $\dfrac{r}{\sqrt{2}}$.

18. $\dfrac{r}{2}$ (r being the radius). 20. 0·882 ; 1·155.

EXERCISE LXVIII.

See ch. XXXVII., § 3.

A.

1. (i) $1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5$;
(ii) $1 + 8i + 28i^2 + 56i^3 + 70i^4 + 56i^5 + 28i^6 + 8i^7 + i^8$;
(iii) $1 + 10i + 45i^2 + 120i^3 + 210i^4 + 270i^5 + 210i^6 + 120i^7 + 45i^8 + 10i^9 + i^{10}$.

2. (i) $c_r = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$;

(ii) $\dfrac{n(n-1)\ldots(n-r+1)}{r!}\,i^r$.

3. c_{n-r}.

4. $1 + (n+1)\,i + \dfrac{(n+1)n}{2!}\,i^2 + \dfrac{(n+1)n(n-1)}{3!}\,i^3 + \ldots$

$$+ \frac{(n+1)n(n-1)(n-2)\ldots(n-r+2)}{r!}\,i^r$$

$$+ \ldots + (n+1)\,i^n + i^{n+1}.$$

7. (i) £1480 4s.; (ii) £1806 2s.; (iii) £2281 1s. 8. £635.

9. (i) $1 - 20x + 180x^2 - 960x^3 + 2960x^4$;
(ii) $256 + 1024x + 1792x^2 + 1792x^3 + 1120x^4$;
(iii) $729a^6 - 5832a^5x + 19440a^4x^2 - 34560a^3x^3 + 34560a^2x^4$;
(iv) $1 - 20x^{\frac{1}{2}} + 190x - 1140x^{\frac{3}{2}} + 4840x^2$;
(v) $128x^7 + 1344x^6 + 6048x^5 + 15120x^4 + 13608x^3$.

10. (i) $-204720x^3$; (ii) $70a^4b^4x^8$; (iii) 70;
(iv) $15a^2b^4$; (v) $-1959552a^5b^5x^5$.

B.

11. n = 2 . . . 1, 2, 3, 4, 5, 6, 7.
n = 3 . . . 1, 3, 6, 10, 15, 21, 28.
n = 4 . . . 1, 4, 10, 20, 35, 56, 84.
n = 5 . . . 1, 5, 15, 35, 70, 126, 210.
n = 6 . . . 1, 6, 21, 56, 126, 252, 462.
n = 7 . . . 1, 7, 28, 84, 210, 462, 924.

12. (i) $1 - 4i + 10i^2 - 20i^3 + 35i^4 - 56i^5 + 84i^6$;
(ii) $1 - 7i + 28i^2 - 84i^3 + 210i^4 - 462i^5 + 924i^6$.

15. $1 - ni + \dfrac{n(n+1)}{2!}\,i^2 - \dfrac{n(n+1)(n+2)}{3!}\,i^3$

$$+ \frac{n(n+1)(n+2)(n+3)}{4!}\,i^4 + \ldots$$

17. 0·08. 20. £822 14s. 21. (i) ·00103; (ii) ·0097.

22. £86 7s. Maximum error = 19s. $4\frac{3}{4}$d.

24. (i) $672x^3$; (ii) $330a^{-12}\,b^7\,x^7$; (iii) $1215 \times 10^{-7}\,x^4$.

C.

28. (i) 1·10302; (ii) 1·144916444;
(iii) $1 + 3{\cdot}2x + 3{\cdot}52x^2 + 1{\cdot}2906x^3$;
(iv) $1 + 4{\cdot}8x + 13{\cdot}92x^2 + 31{\cdot}552x^3$.

31. (i) 51 lbs.; (ii) 22 lbs. 32. 202° C.

D.

34. 2·718.

EXERCISE LXIX.

See ch. XXXVIII., § 4.

A.

4. (i) $5x^4$; (ii) $-5/x^6$; (iii) $28x^6$; (iv) $-40/x^9$;
(v) $20x^3 - 36x^2 + 4x - 10$;
(vi) $6x^5 - 8x^3 + 6x - 6/x^3 + 8/x^5 - 6/x^7$.
5. (v) $60x^2 - 72x + 4$;
(vi) $30x^4 - 24x^2 + 6 + 18/x^4 - 40/x^6 + 42/x^8$.
6. (i) $60x^2$; (ii) $-210/x^8$; (iii) $840x^4$; (iv) $-3600/x^{11}$.
8. (i) $1/2x^{\frac{1}{2}}$; (ii) $2/3x^{\frac{1}{3}}$; (iii) $-3/5x^{\frac{8}{5}}$; (iv) $24x^{1\cdot 4}$; (v) $-7\cdot 2x^{-2\cdot 8}$.
10. (i) $\delta y/\delta x = 3(1+n)^2$; (ii) $\delta y/\delta x = 4/(2-x)^5$;
(iii) $\delta y/\delta x = 7\cdot 2(x - 2\cdot 5)^{\cdot 8}$; (iv) $\delta y/\delta x = 6(2x+3)^2$;
(v) $ma(ax+b)^{m-1}$; (vi) $\delta y/\delta x = -1/3\sqrt{(3x-4)^4}$.

B.

11. (i) $A = 3x^5/5$; (ii) $A = \frac{3}{5}x^{5/3}$; (iii) $3x^{3\cdot 4}/3\cdot 4$.
12. $54991\cdot 2$. 13. (i) $A = \frac{2}{3}(2+x)^{3/2}$; (ii) $A = \frac{3}{10}(2x-1)^{5/3}$;
(iii) $A = \frac{1}{9}/(1-3x)^3$; (iv) $A = -1/7\cdot 2(3x+1)^{3\cdot 4}$

C.

15. (i) $v = -2/(t+1)^2$; $-2, -\frac{1}{2}$; (ii) $v = 9t^{0\cdot 8}$; $0, +9$;
(iii) $v = -10 + 10t^{1\cdot 5}$; $-10, 0$;
(iv) $v = 15/(2t+3)^{7/4}$; $15/3^{7/4}, 15/5^{7/4}$.
16. (i) $s = 2t - \frac{12}{5}t^{5/3}$; $s = -\frac{2}{5}$; (ii) $s = t + t^2 + t^3$; $s = 3$;
(iii) $s = 40[\sqrt{(1+5t)} - 1]$; $20(\sqrt{6} - 1)$;
(iv) $s = 10[(2-3t)^{\frac{2}{3}} - 2^{\frac{2}{3}}]$; $s = 10(1 - 2^{\frac{2}{3}})$.
17. (i) $a = {}^4/(t+1)^3$; (ii) $a = 7\cdot 2t^{-0\cdot 2}$;
(iii) $a = -8/3t^{\frac{1}{3}}$; (iv) $a = 2 + 6t$.
18. (i) $v = 10t - \frac{1}{4}t^2$; (ii) $v = -5t + \frac{2}{3}t^{3/2}$;
(iii) $v = \frac{3}{10}(1+4t)^{2\cdot 5} - \frac{3}{10}$; (iv) $v = (1+2t)^{-2} - 1$.
19. (i) $s = 5t^2 - \frac{1}{12}t^3$; (ii) $s = -\frac{5}{2}t^2 + \frac{4}{15}t^{5/2}$;
(iii) $s = \frac{3}{140}(1+4t)^{3\cdot 5} - \frac{3}{10}t - \frac{3}{140}$;
(iv) $s = -\frac{1}{2}(1+2t)^{-1} - t + \frac{1}{2}$.

D.

21. (i) $\delta y/\delta x = 12x^3$; (ii) $\delta y/\delta x = \frac{2}{3}x^{-\frac{1}{3}}$; (iii) $7\cdot 2x^{1\cdot 4}$.
22. (i) $85°$; (ii) $33°$; (iii) $82°$.
23. (i) 3 ; $(y-3) = 12(x-1)$; (ii) 1 ; $(y-1) = \frac{2}{3}(x-1)$;
(iii) 3 ; $(y-3) = 7\cdot 2(x-1)$.
24. (i) $\frac{3}{4}$; $\frac{1}{4}$; (ii) $-\frac{1}{2}$; $\frac{3}{2}$; (iii) $\frac{7}{12}$; $\frac{5}{12}$.
28. (i) $12(y-3) + (x-1) = 0$; (ii) ${}_3(y-1) + (x-1) = 0$;
(iii) $7\cdot 2(y-3) + (x-1) = 0$.
29. (i) 36 ; (ii) $\frac{2}{3}$; (iii) $21\cdot 6$.

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